Janusz Grabowski

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1. Introduction.

In the usual setting for deformation quantization one looks for a deformed multiplication $*_q$ on the algebra $\mathcal{A} = C^\infty(N)$ of smooth functions on a manifold N of the form

$$u *_q v = uv + qP(u,v) + \mathcal{O}(q^2), \tag{1.1}$$

where P is a given Poisson bracket on $\mathcal{A}$, as q-the deformation parameter goes to 0. In application to quantum mechanics [BFF] N will be the phase space of a classical mechanical system endowed with its symplectic structure and the corresponding Poisson bracket.

Recall that the Poisson bracket P in general is a Lie bracket being a biderivative:

$$P(u, vw) = P(u,v)w + vP(u,w).$$

Poisson brackets on $\mathcal{A}$ are in a one-one correspondence with Poisson structures on N, i.e. those bivector fields $P \in \Gamma(\Lambda^2(TN))$ which satisfy $[P,P] = 0$, where $[\cdot,\cdot]$ stands for the Schouten bracket, so we shall use both notions interchangeably. The "formal" version of (1.1) reads

$$S(u,v) := u *_q v = uv + qP(u,v) + \sum_{k=2}^{\infty} q^k P_k(u,v), \tag{1.2}$$

where one wants usually the operators P_k to be bidifferential and vanishing on constants, symmetric for k even and skewsymmetric for k odd. In this case $*_q$ is called *star product* for P and it is proven to exist by De Wilde and Lecomte [DWL] for symplectic Poisson brackets. The question of existence of a star product for arbitrary P remains open and only partial results are known (cf. [Gr2]). In this note we answer it affirmatively in the case of the simplest but in general non-symplectic Poisson structures P as those of the form $P = X \wedge Y$, where $X, Y \in \mathcal{X}(N)$ are vector fields on N. Since vanishing of the Schouten bracket $[P,P]$ is in this case equivalent to $X \wedge Y \wedge [X,Y] = 0$, we consider Poisson structures of the form $P = X \wedge Y$ with $[X,Y] = uX + vY$ for some $u, v \in \mathcal{A}$.

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Example. Let P be the Poisson bracket on the sphere S^3 associated with the Woronowicz' [W] $SU(2)$ group. In global coordinates $(a,b,x,y) \in \mathbf{R}^4$, $a^2+b^2+x^2+y^2=1$ on $SU(2) \simeq S^3 \subset \mathbf{R}^4$ (cf. [Gr1]):

$$P(x,a) = -xb, \quad P(x,b) = xa, \quad P(x,y) = 0,$$

$$P(y,a) = -yb, \quad P(y,b) = ya, \quad P(b,a) = x^2+y^2.$$

The corresponding Poisson structure can be written in the form $P = X \wedge Y$, where

$$X = bZ - \frac{\partial}{\partial b}, \quad Y = aZ - \frac{\partial}{\partial a},$$

and

$$Z = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

It is easy to see that X and Y are really tangent to the sphere and that $[X,Y] = aX - bY$.

2. Differentiable Hochschild cohomology.

Looking for star-products, it is convenient to use the language of the Gerstenhaber bracket $[\cdot,\cdot]_G$ [Ge] on the space $M(V)$ of all multilinear mappings of a vector space V into itself. This space is naturally graded and the Gerstenhaber bracket makes it into a graded Lie algebra with $(n+1)$-linear mappings being of degree n. Note that this structure was rediscovered and used by De Wilde and Lecomte [DWL].

Recall that for bilinear $A,B : V \times V \longrightarrow V$ we have in particular $[A,B]_G = i(B)a + i(A)B$, where

$$i(B)A(x,y,z) = A(B(x,y),z) - A(x,B(y,z)).$$

Hence $[A,A]_G = 0$ if and only if A is an associative operation. The associativity condition for star-product

$$S = \sum_{k=0}^{\infty} q^k P_k,$$

where P_0 stands for the standard multiplication in $\mathcal{A}$ and P_1 is the given Poisson bracket, may be therefore written as $[S,S]_G = 0$ or, equivalently, as

$$(E_k) \qquad \sum_{i+j=k} [P_i,P_j]_G = 0,$$

where $k = 0,1,2,\ldots$. The equation (E_0) is simply the associativity of the standard product and (E_1) easily follows from the fact that the Poisson structure is a biderivation. To construct the star product inductively, consider $S_n = \sum_{i=0}^{n} q^i P_i$. We say that S_n is *associative of order* n if $[S_n,S_n]_G = \mathcal{O}(q^{n+1})$, i.e. (E_k) is satisfied for $k = 0,1,\ldots,n$. Having a star-product S_n of order n we look for P_{n+1} such that (E_{n+1}) holds. It is easy to see that $[P_0,P_j]_G$ is exactly δP_j, the Hochschild coboundary of P_j, so (E_{n+1}) may be written as $2\delta P_{n+1} = J_{n+1}$, where

$$J_{n+1} = \sum_{\substack{i+j=n+1\\ i,j>0}} [P_i,P_j]_G.$$

Since $[S_n,S_n]_G = q^{n+1}J_{n+1} + \mathcal{O}(q^{n+2})$ and due to the graded Jacobi identity $[S_n,[S_n,S_n]_G]_G = 0$, we get $\delta J_{n+1} = [P_0,J_{n+1}]_G = 0$, so we know that J_{n+1} is a Hochschild cocycle and we only need J_{n+1} to be coboundary. Hence the obstruction to construct the star-product inductively is the 3th Hochschild cohomology. Assuming the operators P_k being differential, we work in the differentiable Hochschild cohomology $HH^3_{diff}(\mathcal{A})$.

Theorem 1. (Vey [V], Cahen, Gutt, De Wilde [CDW])

$$HH^p_{diff}(\mathcal{A}) \simeq \Gamma(\Lambda^p(TN)).$$

To prove the above theorem, one localizes (observe that $\delta(uP) = u\delta(P)$) and uses the fact that locally the Hochschild complex of differential operators on $\mathcal{A}$ is naturally isomorphic to the complex $(V_r^*(\mathcal{A}), \delta)$, where

$$V_r^p(\mathcal{A}) = \mathcal{A} \otimes \underbrace{V_r \otimes \ldots \otimes V_r}_{p-times},$$

$V_r = \mathbf{R}[x_1, \ldots, x_r]$ is the ring of polynomials in $r = dim(N)$ variables, and the Hochschild coboundary operator has the form

$$\delta(a \otimes w_0 \otimes \ldots \otimes w_{p-1}) = a \otimes (\sum_{i=0}^{p-1} w_0 \otimes \ldots \otimes c(w_i) \otimes \ldots \otimes w_{p-1})$$

with $c : V_r \longrightarrow V_r \otimes V_r$ being defined by $c(w) = \Delta w - 1 \otimes w - w \otimes 1$ for the standard coassociative coproduct Δ in V_r regarded as the symmetric algebra – the universal enveloping algebra of the commutative r-dimensional Lie algebra. In particular, $c(x_i) = 0$ and $c(x_i x_j) = x_i \otimes x_j + x_j \otimes x_i$. Note that the algebra $\mathcal{A}$ is in this case the algebra of smooth functions on the corresponding neighbourhood, but the complex makes sense for arbitrary algebra. The algebraic result which implies Theorem 1 and which we shall use later on is the following.

Theorem 2.

$$H^p(V_r^*(\mathcal{A})) \simeq \mathcal{A} \otimes \Lambda^p(\mathbf{R}^r).$$

3. Special Hochschild cohomology and quantization.

We shall consider multidifferential operators on the algebra $\mathcal{A} = C^\infty(N)$ generated by given vector fields. Our main observation is the following.

Theorem 3 *Let $D_1, \ldots, D_r \in \mathcal{X}(N)$ be smooth vector fields on N linearly independent on a dense subset Ω of N and such that the $\mathcal{A}$-module $\mathcal{L}$ which they generate is a Lie subalgebra of $\mathcal{X}(N)$. Then*

a) the algebra $\mathcal{U}$ generated by $\mathcal{L}$ in the algebra $Diff(\mathcal{A})$ of linear differential operators on $\mathcal{A}$ is a free $\mathcal{A}$-module isomorphic to $V_r^1(\mathcal{A})$;

b) there is an embedding

$$j : \mathcal{U}^p = \underbrace{\mathcal{U} \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \mathcal{U}}_{p-times} \longrightarrow Diff_p(\mathcal{A})$$

of $\mathcal{U}^p$ into the space of p-linear differential operators on $\mathcal{A}$ such that $\mathcal{U}^* := \bigoplus_{p=0}^\infty j(\mathcal{U}^p)$ is a subcomplex of the differentiable Hochschild complex $Diff_*(\mathcal{A})$ invariant with respect to the Gerstenhaber bracket;

c) The complex $(\mathcal{U}^*, \delta)$ is isomorphic to the complex $(V_r^*(\mathcal{A}), \delta)$.

Note that the part a) of the above theorem may be regarded as a version of the Poincaré-Birkhoff-Witt theorem in spite of the fact that the Lie bracket is not $\mathcal{A}$-linear, since $[D_i, fD_j] = f[D_i, D_j] + D_i(f)D_j$.

Proof. a) We claim that

$$\{D^\alpha : \alpha = (\alpha_1, \ldots, \alpha_r), \alpha_i = 0, 1, \ldots, \; i = 1, \ldots, r\}$$

is a basis of $\mathcal{U}$ over $\mathcal{A}$, where $D^\alpha = D_1^{\alpha_1} \circ \ldots \circ D_r^{\alpha_r}$. As in the classical Poincaré-Birkhoff-Witt theorem, it is obvious that it is a set of generators, so suppose that $\sum_{|\alpha|\le k} c_\alpha D^\alpha = 0$ for some $c_\alpha \in \mathcal{A}$. It suffices to show now that $c_\alpha = 0$ for $|\alpha| = k$. Take $\omega \in \Omega$. Since our vector fields are linearly independent at ω, there are (globally defined!) $z_1, \ldots z_{dim(N)} \in \mathcal{A}$ vanishing at ω and defining such a coordinate system in a neighbourhood of ω that $D_i(\omega) = \partial_{z_i}$, $i = 1, \ldots, r$. Hence $D_i = \partial_{z_i} + D_i'$, where $D_i'(\omega) = 0$ and $D^\alpha = \partial^\alpha + Y_\alpha + Z_\alpha$, where $\partial^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_r}^{\alpha_r}$, Y_α is a differential operator vanishing at ω, and Z_α is a differential operator of order $<|\alpha|$. For $z^\beta = z_1^{\beta_1} \cdots z_r^{\beta_r}$, where $|\beta| = |\alpha|$, we have then $D^\alpha(z^\beta)(\omega) = 0$ if $\alpha \neq \beta$ and $D^\alpha(z^\alpha)(\omega) = \alpha!$, where $\alpha! = \alpha_1! \cdots \alpha_r!$. Thus for $|\beta| = k$ we have

$$\sum_{|\alpha|\le k} c_\alpha D^\alpha(z^\beta)(\omega) = \beta! c_\beta(\omega) = 0,$$

so all functions c_α vanish on the dense subset Ω and hence on the whole N.

b) We define

$$j(u_1 \otimes \ldots \otimes u_p)(f_1, \ldots, f_p) = u_1(f_1) \cdots u_p(f_p)$$

and it is easy to see that j is a well-defined map. To prove it injectivity it suffices to show that if

$$\sum_{|\alpha^1| + \ldots + |\alpha^p| \le k} c_{\alpha^1 \ldots \alpha^p} D^{\alpha^1}(f_1) \cdots D^{\alpha^p}(f_p) = 0$$

for all $f_1, \ldots, f_p \in \mathcal{A}$ then all functions $c_{\alpha^1 \ldots \alpha^p} \in \mathcal{A}$ vanish, what easily follows from a) by induction.

Since our vector fields are derivations of $\mathcal{A}$,

$$D^\alpha(f_1 \cdots f_p) = \sum_{\beta^1 + \ldots + \beta^p = \alpha} \frac{\alpha!}{\beta^1! \cdots \beta^p!} D^{\beta^1}(f_1) \cdots D^{\beta^p}(f_p).$$

This implies that $i(j(u_1 \otimes \ldots \otimes u_p))u \in \mathcal{U}^*$ for any $u, u_1, \ldots, u_p \in \mathcal{U}$ and finally that $\mathcal{U}^* \subset Diff_*(\mathcal{A})$ is closed with respect to the Gerstenhaber bracket. Since the standard multiplication in $\mathcal{A}$ may be written as $j(1 \otimes 1)$, and the Hochschild coboundary operator is (up to a sign) the Gerstenhaber bracket with the multiplication, $\mathcal{U}^*$ is a subcomplex of $Diff_*(\mathcal{A})$ (cf. [Gr2]).

c) The obvious computations show that the identification of D^α with x^α leads to the identification of Hochschild complexes $(\mathcal{U}^*, \delta)$ and $(V_r^*(\mathcal{A}), \delta)$.

Corollary. $H^p(\mathcal{U}^*, \delta) = \mathcal{A} \otimes \Lambda^p(\mathbf{R}^r)$.

Theorem 4. *Every Poisson structure P on a manifold N of the form $P = X \wedge Y$, where X, Y are vector fields on N satisfying $[X, Y] = uX + vY$ for some smooth functions u and v admits a star-product.*

Proof. Since X and Y are linearly independent on a dense subset of N and the $\mathcal{A} = C^\infty(N)$-module they generate is a Lie algebra, due to Theorem 3 they generate an algebra of differential operators $\mathcal{U}$ and the Gerstenhaber subalgebra of multilinear differential operators $\mathcal{U}^*$ with the Hochschild cohomology (or, perhaps better to say, co-Hochschild homology) $H^p(\mathcal{U}^*, \delta) \simeq \mathcal{A} \otimes \Lambda^p(\mathbf{R}^2)$. In particular, $H^3(\mathcal{U}^*) = 0$. Constructing a star-product inductively, we start we the standard multiplication P_0 and the Poisson structure $P_1 = P$ which belong to $\mathcal{U}^*$. Inductively, the Hochschild cocycles

$$J_n = \sum_{\substack{i+j=n \\ i,j>0}} [P_i, P_j]_G$$

belong to $\mathcal{U}^*$ which is closed with respect to the Gerstenhaber bracket and we must look for $P_n \in j(\mathcal{U}^2)$ such that $2\delta(P_n) = J_n$, what is always possible because of vanishing of $H^3(\mathcal{U}^*)$.

In particular, the Lie-Poisson structure of the group $SU(2)$ described in our Example admits a star-product. Probably one of them gives the product of Woronowicz, but it is hard to be seen, since our procedure is not constructive nor unique.

Remark. Note that all our considerations remain true in the real-analytic case as well.

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Janusz Grabowski

Institute of Mathematics
University of Warsaw
ul. Banacha 2
PL 02-097 Warsaw, Poland

e-mail: jagrab@mimuw.edu.pl