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Symplectic models of groups with noncommutative spaces

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SYMPLECTIC MODELS OF GROUPS WITH NONCOMMUTATIVE SPACES *

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1 Introduction: generalizing groups

1.1 Quantum spaces

The Gelfand-Naimark duality states a natural anti-equivalence between the category of locally compact topological spaces and the category of commutative C*-algebras. By definition, a *locally compact quantum space* or a C*-*space* is an object of the category formally dual to the category of all (not necessarily commutative) C*-algebras (with morphisms defined in [10]).

This is an example of a natural embedding of a category of spaces of certain type into a larger category. The trick consists in the passage from a space Λ to an appropriate algebra A_Λ of functions on Λ. The algebra A_Λ should contain the same information as the space Λ (in particular, it should be possible to recover Λ from A_Λ). The class of algebras corresponding to all spaces Λ (of given type) should be characterized independently, i.e. without referring to the construction of A_Λ from a particular Λ. Then one can enlarge given category by removing the commutativity condition. The theory of spaces generalized in such a way is known as *non-commutative geometry.* The generalized spaces are said to be *non-commutative spaces* or *quantum spaces.*

1.2 Quantum groups

One can investigate the notion of a group structure on a quantum space. A quantum space being represented by an algebra A, an associative multiplication law has to be represented by a co-associative morphism Δ from A to $A \otimes A$ (suitable tensor product), called *comultiplication.* A pair (A, Δ) represents in

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this case a *quantum semi-group* (of the type of the considered quantum spaces). *Quantum groups* are quantum semi-groups satisfying additional conditions replacing the usual condition of the existence of the neutral element and the group inverse. Unfortunately, the technical form of those conditions depends on the class of underlying quantum spaces. From the point of view of algebraic groups, where the natural algebra of functions is the algebra of polynomials (say, in matrix elements), it is natural to require (A, Δ) to be a Hopf *-algebra (see [5, 13] for the definition), i.e. to require the existence of a *counit* and a *co-inverse* (*antipode*). In the context of C*-spaces, the required conditions are formulated somewhat differently ([1, 11]; in fact, for non-compact case the proper conditions are still to be found), but are closely related to the previous ones and play a similar role. It is convenient to call pairs satisfying those conditions *Hopf* C**-algebras.* We summarize the approach based on C*-algebras in the following example.

EXAMPLE 1. C*-*groups*: groups with C*-*spaces.* They arise in the following generalization scheme.

$$\begin{array}{ccccc} \textit{Locally compact group} & & \textit{commutative} & & \textit{Hopf } C^*\textit{-algebra} \\ G = (\Lambda, m) & \mapsto & \textit{Hopf } C^*\textit{-algebra} & \dashrightarrow & (A, \Delta) \\ & & (A_\Lambda, \Delta_m) & & \end{array}$$

(Δ_m is the comultiplication corresponding to the group multiplication m in G).

1.3 Symmetries in physics

Groups with non-commutative spaces can be used as symmetry groups in physics. The following table shows schematically the difference between classical and quantum systems in this respect.

	symmetries	SYSTEMS: classical (P)	SYSTEMS: quantum (H)
G	usual	hamiltonian actions	unitary rep's
R	$G = (\Lambda, m)$	(G on P)	(G in H)
O	C*-	–	unitary u in $A \otimes \mathcal{K}(H)$
U	$G \leftrightarrow (A, \Delta)$	–	$(\Delta \otimes \mathrm{id})u = u_{13}u_{23}$
P	? -	?	–
S	$G \leftrightarrow$?	?	–

Here P denotes a symplectic manifold (describing a classical system), H denotes a Hilbert space (describing a quantum system) and $\mathcal{K}(H)$ is the algebra of compact operators in H.

A family of unitary operators in H, $\lambda \mapsto u_\lambda$, can be identified as a unitary element u of the multiplier algebra [10] of $A_\Lambda \otimes \mathcal{K}(H)$ (in the above table we should also use the multiplier algebra; we omit it for simplicity). The notion of a *quantum family of unitaries* in H is a straightforward generalization of the usual family of unitaries: we simply replace A_Λ by an arbitrary (non-commutative) C*-algebra A. This possibility allows to define a unitary representation of a C*-group without difficulty.

Note that only the nature of the index of the family is changed, the nature of the 'second leg' of the family – related to $\mathcal{K}(H)$ – being unchanged. No such comfort for a family of symplectomorphisms is possible: quantum family of symplectomorphisms is an absurd. If we really want to consider 'non-commutative families of symplectomorphisms', we have to modify the non-commutative geometry. The question marks in the above table stand for the solution of this problem. We state the problem as follows.

Question: What structure generalizes spaces (and groups) in a way which is compatible with classical systems?

In the next section we present a solution of this problem. We obtain a generalization of Lie groups suitable to describe symmetries of classical systems (and not suitable for quantum systems — the right bottom corner of the table will be empty).

2 Symplectic non-commutative geometry

In this section we modify the idea of non-commutative geometry, replacing function algebras by structures better adapted to classical physical systems. Non-commutative geometry introduces the notion of a non-commutative space. Such space has no direct meaning (as a collection of points). Instead,

what makes sense is the algebra of "functions on the space"

(one can not abandon " " here). The idea of the *symplectic non-commutative geometry* is similar:

what makes sense is the "cotangent bundle of a manifold"

and not the manifold itself.

2.1 The cotangent bundle algebra

Consider first the Gelfand-Naimark correspondence $\Lambda \mapsto A_\Lambda$ between locally compact spaces and commutative C*-algebras. To any continuous map $\phi\colon \Lambda \longrightarrow \Lambda'$

there corresponds $\phi^*: A_{\Lambda'} \to A_\Lambda$ given by the **pullback** operation $(\phi^* f)(\lambda) = f(\phi(\lambda))$ (for non-compact Λ's, arrows between A_Λ's are understood in the sense of [10]). If a structure on Λ is expressed by a continuous map, it can be translated into a structure on A_Λ using the pullback. An example is provided by the comultiplication: $\Delta_m = m^*$. The structure of A_Λ itself is defined by pullback as follows. With any Λ there are associated two characteristic mappings

– the *diagonal* map $d: \Lambda \to \Lambda \times \Lambda$, and

– the *constant* map $c: \Lambda \to \Lambda_0 = \{1\}$ (one-point set).

They define by pullback, respectively,

– the *multiplication* $\bullet = d^*: A_\Lambda \otimes A_\Lambda \to A_\Lambda$, and

– the *unit* $\mathbf{1} = c^*: \mathbb{C} \to A_\Lambda$

of the algebra A_Λ.

Let Λ, Λ' be manifolds. **Pullback of covectors** by a (smooth) map $\phi: \Lambda \to \Lambda'$ is defined as a **relation** $\mathrm{T}^*\phi$ from $\mathrm{T}^*\Lambda'$ to $\mathrm{T}^*\Lambda$ that relates $x \in \mathrm{T}^*{}_\lambda\Lambda$ to $x' \in \mathrm{T}^*{}_{\lambda'}\Lambda'$ if and only if $\lambda' = \phi(\lambda)$ and $< x', (T\phi)v >=< x, v >$ for all $v \in \mathrm{T}_\lambda\Lambda$.

Remark. Relation $\mathrm{T}^*\phi$ is symplectic [12] (or *canonical* [7, 8]).

NOTATION: We shall write $r: X -\!\rhd Y$ instead of "r is a relation from X to Y" (the special arrow is used to distinguish relations from maps).

Let Λ be a manifold and $X = \mathrm{T}^*\Lambda$ its cotangent bundle. Pullback by d defines the *multiplication relation*

$$\bullet = \mathrm{T}^*d: X \times X -\!\rhd X$$

and c defines the *unit relation*

$$\mathbf{1} = \mathrm{T}^*c: \{1\} -\!\rhd X.$$

We have

$$x_1 \bullet x_2 = \begin{cases} \emptyset & \lambda(x_1) \neq \lambda(x_2) \\ x_1 + x_2 & \text{otherwise} \end{cases}$$

and the image of $\mathbf{1}$ is the set of all null covectors in X.

The multiplication relation is *associative* (since d is co-associative):

$$\bullet(\bullet \times \mathrm{id}) = \bullet(\mathrm{id} \times \bullet) \tag{1}$$

and $\mathbf{1}$ obeys the *unit property* (since c is a co-unit for d):

$$\bullet(\mathbf{1} \times \mathrm{id}) = \mathrm{id} = \bullet(\mathrm{id} \times \mathbf{1}). \tag{2}$$

We have also the *star operation* $*: X \to X$, given by $x \mapsto -x$, which is anti-symplectic and satisfies

$$** = \mathrm{id} \qquad \text{and} \qquad (x_1 \bullet x_2)^* = x_2^* \bullet x_1^* \ \text{ for } x_1, x_2 \in X \tag{3}$$

(i.e. $*$ is involutive and anti-multiplicative).

DEFINITION. A *symplectic *-algebra* (with unit) is a quadruple $(X; \bullet, \mathbf{1}, *)$, where

X is a symplectic manifold

$\bullet: X \times X \rightarrowtriangle X$, $\mathbf{1}: \{1\} \rightarrowtriangle X$ are symplectic relations satisfying (1), (2)

$*: X \to X$ is a relation satisfying (3).

The *cotangent bundle algebra* $\mathbb{A}_\Lambda = (\mathrm{T}^*\Lambda; \mathrm{T}^*d, \mathrm{T}^*c, -)$ is therefore a special example of a symplectic *-algebra. In this example, the multiplication is commutative.

2.2 Symplectic non-commutative possibility

The multiplication in a symplectic *-algebra need not to be commutative.

EXAMPLE 2. Let P be a symplectic manifold. We denote by $\bar{P}$ the same manifold with the opposite symplectic form. The quadruple $\mathrm{End}(P) = (P \times \bar{P}; \circ, \mathrm{id}, \dagger)$, where $(p_1, p_2)^\dagger = (p_2, p_1)$ and

$$(p_1, p_2) \circ (p_3, p_4) = \begin{cases} \emptyset & p_2 \neq p_3 \\ (p_1, p_4) & \text{otherwise} \end{cases}$$

is a symplectic *-algebra.

$\mathbb{A}_\Lambda$ and $\mathrm{End}(P)$ are in fact examples of S*-algebras: objects introduced in [12] (cf. also [2]). A *S*-algebra* is a symplectic *-algebra satisfying some additional (regularity and 'positivity') conditions. S*-algebras turn out to play the role analogical to C*-algebras and can be used to generalize ordinary manifolds. We are thus led to S*-spaces and S*-groups.

DEFINITION. S*-*groups*: groups with S*-*spaces* (see [12] for a rigorous definition; unlike in Example 1, the definition introduces the co-unit ϵ and the co-inverse κ). We have then the following scheme of generalization:

$$\begin{matrix} \textit{Lie group} \\ G = (\Lambda, m) \end{matrix} \ \mapsto \ \begin{matrix} \textit{commut. Hopf S}^*\textit{-algebra} \\ (\mathbb{A}_\Lambda, \Delta_m) \end{matrix} \ \dashrightarrow \ \begin{matrix} \textit{Hopf S}^*\textit{-algebra} \\ (\mathbb{A}, \Delta) \end{matrix}$$

(cf. Example 1; here $\Delta_m = \mathrm{T}^*m$).

Remarks:

1. A 'hamiltonian action' of a S*-group on a symplectic manifold P can be now. defined in the same way as a unitary representation of a C*-group in a Hilbert space H (one replaces A by $\mathbb{A}$, Δ by Δ and $\mathcal{K}(H)$ by $\mathrm{End}(P)$). A *unitary element* of a S*-algebra $\mathbb{A} = (X; \bullet, \mathbf{1}, *)$ is a symplectic relation $v: \{1\} \multimap X$ such that

$$\bullet(v \times v^*) = \bullet(v^* \times v) = \mathbf{1},$$

both compostions here being assumed to be transversal in the sense of [12] (it turns out that unitary elements are exactly smooth lagrangian bi-sections of [2]).

2. Already on this level, one can expect S*-groups to be useful for studying C*-groups. Namely, one can:

– try to replace the symplectic relations constituing a S*-algebra by linear operators (geometric quantization) to obtain a Hopf C*-algebra,

– study the properties of C*-groups by looking at the properties of S*-groups.

3. The *dual* of $G \leftrightarrow (\mathbb{A}, \Delta) = ((X; \bullet, \mathbf{1}, *), \Delta, \epsilon, \kappa)$ is obtained by applying the transposition of relations, $r \mapsto r^T$, to all ingredients:

$$((X; \Delta^T, \epsilon^T, *\kappa), \bullet^T, \mathbf{1}^T, \kappa) \leftrightarrow \hat{G}$$

(ϵ is the *co-unit* and κ is the *co-inverse*).

3 More about S*-spaces: twisted cotangent bundles

Let $\mathbb{A} = (X; \bullet, \mathbf{1}, *)$ be a S*-algebra. We set $I = \mathrm{im}\,\mathbf{1} \subset X$ (I is called the *set of units*). By (2), $I \bullet x = x = x \bullet I$ for each $x \in X$. It follows that for each $x \in X$ there is exactly one $a = i_L(x) \in I$ such that $a \bullet x \neq \emptyset$ and exactly one $b = i_R(x) \in I$ such that $x \bullet b \neq \emptyset$. (Indeed, the existence is obvious. If $a' \in I$ is such that $a' \bullet x \neq \emptyset$ then $a \bullet a' \neq \emptyset$, hence $a = a \bullet a' = a'$.) The maps $i_L, i_R: X \to I$ are said to be the *left* and the *right projection*, respectively. By regularity, these mappings turn out to be smooth submersions.

One can show that the Poisson bracket of two functions constant on th fibers of the left (right) projection is again a function constant on these fibers. Therefore, the left (right) projection defines a Poisson bracket π_L (π_R) on I. The two Poisson brackets differ only by the sign: $\pi_L + \pi_R = 0$. If $\mathbb{A}$ is commutative, both projections coincide and $\pi_L = \pi_R = 0$.

Morphisms of S*-algebras [12] turn out to have the structure similar to a pullback of covectors. Any morphism from a S*-algebra $\mathbb{A}'$ to a S*-algebra $\mathbb{A}$ induces a map $\phi : I \rightarrow I'$, called the *base map* of the morphism. The base map preserves the Poisson structure and (under certain conditions [12]) determines the morphism completely.

Poisson manifolds (in fact not all) can be considered as 'infinitesimal version' of S*-spaces. To some extent, both versions are equivalent.

Corollary: Certain Poisson manifolds are infinitesimal versions of S*-spaces. (Cf. also *integrable* Poisson manifolds in [9].)

4 More about S*-groups: Manin groups

Consider a Lie group $G = (\Lambda, m)$. $T^*\Lambda$ carries two structures of S*-algebras:

– the *space* S*-algebra $\mathbb{A}_\Lambda$ (i.e. the cotangent bundle algebra), with **one** projection on $I \cong \Lambda$, and

– the *group* S*-algebra $(T^*\Lambda, {\Delta_m}^T)$, with two projections on $J = \mathrm{im}\ \epsilon$ (the projections coincide with the right and left translation to the group unit).

In a general S*-group we have **two** projections on I and two projections on J.

I is a group under the base map of Δ and J is a group under the base map of $\bullet^T$. Moreover, it turns out that there is a unique group structure on X such that I, J are subgroups and the projections act along the corresponding cosets. Each element $x \in X$ is uniquely decomposed as a product of two elements: one from I and the other one from J (in two possible orderings):

$$x = i_L(x) j_R(x) = j_L(x) i_R(x).$$

The existence of the decomposition corresponds to the condition $IJ = X$ and the uniqueness corresponds to $I \cap J = \{\text{one point}\}$. (Any group X with two subgroups I, J satisfying these conditions is said to be a *double group*.) At each point $x \in X$ we have two decompositions of the tangent space: the *left* (*right*) *decomposition* onto subspaces prallel to the left (right) cosets. Denote by ρ_L (resp. ρ_R) the reflection in the tangent space corresponding to the left (right) decomposition and set $\rho = \frac{1}{2}(\rho_L + \rho_R)$. It turns out that the symplectic form on X, modified by ρ defines an invariant non-degenerate scalar product, vanishing on vectors tangent to I and vectors tangent to J. A double Lie group with such a scalar product is said to be a *Manin group*. It is proved in [12] that the correspondence between S*-groups and Manin groups is one-to-one.

S*-groups have another (but related) important connection: with Poisson groups [3, 6, 4]. (Let us remind, that a *Poisson (Lie) group* is a pair (G, π), where G is a Lie group and π is a Poisson bracket on G such that the group multiplication preserves the Poisson structure.) The set of units I as above is equipped with a Poisson bracket π (we choose $\pi = \pi_L$). On the other hand, I is a group whose multiplication is the base map of $\triangle$. It follows that the multiplication map, $m: I \times I \to I$ preserves the Poisson structure, hence $((I, m), \pi)$ is a Poisson group.

5 Poisson groups, Lie bialgebras and S*-groups

Let (G, π) be a Poisson group. The Poisson structure is here represented by the bi-vector field π on G. The compatibility of the group multiplication with π can be written as follows:

$$\pi(gh) = g\pi(h) + \pi(g)h \tag{4}$$

for $g, h \in G$. Here $g\pi(h)$ $(\pi(g)h)$ means the left (right) translation of $\pi(h)$ by g ($\pi(g)$ by h). Let $\mathfrak{g}$ denote the Lie algebra of G. Since $\pi(e) = 0$ (here e is the group unit), the linearization $\delta = d\pi(e): \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ of π at e is well defined. We have the following properties:

1. δ is a linear Poisson bracket on $\mathfrak{g}$,
2. δ is a 1–cocycle on $\mathfrak{g}$.

Any pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra and $\delta: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ has the above properties, is said to be a *Lie bialgebra.* There is a one-to-one correspondence between Lie bialgebras and connected simply connected Poisson groups. Lie bialgebras are infinitesimal versions of Poisson groups.

The *dual* Lie bialgebra is obtained by dualizing δ and the commutator $[\ ,\ ]$:

1° $\delta^*: \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$,

2° $[\ ,\ ]^*: \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ turns out to be a 1–cocycle on $\mathfrak{g}^*$.

In order to prove 2° one usually sets up a 1–1 correspondence between Lie bialgebras and so called Manin triples, using the following theorem [4].

Theorem. Let $\mathfrak{g}$, $\mathfrak{g}^*$ be two mutually dual spaces equipped with Lie brackets. Denote by $\delta: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ the map dual to the bracket on $\mathfrak{g}^*$ and set $\Xi = \mathfrak{g} \oplus \mathfrak{g}^*$. There exists exactly one bilinear skew-symmetric operation $[\cdot, \cdot]: \Xi \times \Xi \to \Xi$ extending the given Lie brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ and such that the scalar product

$$< v \oplus \alpha, v' \oplus \alpha' >= \alpha(v') + \alpha'(v)$$

is invariant (with respect to $[\cdot,\cdot]$). Moreover, $[\cdot,\cdot]$ is a Lie bracket if and only if $(\mathfrak{g}, \delta)$ is a Lie bialgebra.

The system $(\Xi; \mathfrak{g}, \mathfrak{g}^*, < \cdot, \cdot >)$ is said to be the *Manin triple* (or *Manin algebra*) corresponding to Lie bialgebra $(\mathfrak{g}, \delta)$. A Manin algebra can be used as a starting point to construct a Manin group (hence an example of a S*-group). To this end we consider the group X corresponding to Ξ and also the subgroups G, G^* in X corresponding to $\mathfrak{g}$, $\mathfrak{g}^*$. If $GG^* = X$ and $G \cap G^* = \{\text{one point}\}$, we obtain a Manin group. In general however these conditions are not satisfied.

Corollary: Lie bialgebras are double infinitesimal versions of S*-groups. Not every Lie bialgebra (not every Poisson group) produces an example of a S*-group (cf. also [14]).

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