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VECTOR-DERIVED MANIFOLDS

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1. Introduction and preliminaries

In recent years, Poisson manifolds have become a main subject of interest of a large group of mathematicians and physicists working in theoretical mechanics and differential geometry. Such manifolds were for the first time considered by Lie in some non-explicit form (see [11]) and introduced by Lichnerowicz [8] as a generalization of symplectic manifolds which play a crucial rôle in theoretical mechanics. Recently, one can observe an increasing significance of the concept of Poisson manifolds in connection with the development of the theories of Poisson-Lie groups and of quantum groups.

By definition, a Poisson manifold is a pair (M,F) where M is a differentiable manifold of finite dimension and F is a bicontravariant tensor field on M such that the Schouten-Nijenhuis bracket [F,F] equals 0. The condition [F,F] = 0 is regarded to be an integrable condition for F, which allows to introduce some differential operators for a complex of cochains of antisymmetric vector fields on M (see [8]) and a complex of chains of differential forms on M (see [7] and [3]). Moreover, this condition is equivalent to the fact that the linear space $C^{\infty}(M)$ is a Lie algebra under the Poisson bracket. One can ask whether there exists a symmetric analogue of the notion of Poisson manifold, as well as a complex one. An answer to this question is presented in [10] where the problem of finding an integrable condition for the corresponding bicontravariant tensor field is the most interesting. On the other hand, it is also interesting to consider such analogues together with the notion of Poisson manifold from the one point of view. In the case of real manifolds, this is the aim of the paper. More precisely, we introduce the concept of a vector-derived manifold as a

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generalization of that of Poisson manifold. This generalization contains a symmetric analogue of the notion of Poisson manifold (Section 2), however, we do not take into consideration any corresponding integrable condition. For this reason our definition of such an analogue is based on the notion of Poisson manifold.

In this paper we accept the terminology and notation from [6] unless otherwise stated. By a differentiable manifold we shall always mean a paracompact C^∞ manifold of finite dimension. If M is a differentiable manifold, we adopt the following notations:

$C^\infty(M)$: the algebra of all real smooth functions on M under the pointwise operations;

TM: the tangent manifold (bundle) of M;

T_xM: the tangent vector space of M at x;

T^*M: the cotangent manifold (bundle) of M;

T^*_xM: the cotangent vector space of M at x;

$\mathcal{X}(M)$: the $C^\infty(M)$-module of all vector fields on M;

$\mathcal{D}(M)$: the $C^\infty(M)$-module of all differential forms on M.

The module $\mathcal{X}(M)$ ($\mathcal{D}(M)$) will be regarded as the $C^\infty(M)$-module of all smooth sections of the tangent (cotangent) bundle TM (T^*M). If $X \in \mathcal{X}(M)$ ($\xi \in \mathcal{D}(M)$), then by X_x (ξ_x) we denote the value of X (ξ) at $x \in M$. If in addition $\alpha \in C^\infty(M)$, then by $X(\alpha)$ ($d\alpha$) we denote the action of X on α (the differential of α). Moreover, we set $d_x\alpha = (d\alpha)_x$ for $x \in M$. For any $X \in \mathcal{X}(M)$ and $\xi \in \mathcal{D}(M)$ we denote by $\langle\xi,X\rangle$ the differential pairing of M which defines the $C^\infty(M)$-bilinear map from $\mathcal{D}(M) \times \mathcal{X}(M)$ to $C^\infty(M)$. This pairing is uniquely defined by the condition $\langle d\alpha,X\rangle = X(\alpha)$ for all $\alpha \in C^\infty(M)$, $X \in \mathcal{X}(M)$.

For any $X,Y \in \mathcal{X}(M)$ we denote by $[X,Y]$ the usual bracket of vector fields on M. It is known that the linear space $\mathcal{X}(M)$ equipped with this bracket is a Lie algebra called the *Lie algebra* of *vector fields* on M and denoted by $(\mathcal{X}(M),[\cdot,\cdot])$ or by $\mathcal{X}(M)$, as well.

Let J be an almost complex structure on M, that is, J is a module automorphism of $\mathcal{X}(M)$ satisfying $J^2 = -1$ where 1 denotes the identity map of $\mathcal{X}(M)$. We denote by J^* the module automorphism of $\mathcal{D}(M)$ adjoint to J, which means that $\langle J^*\xi,X\rangle = \langle\xi,JX\rangle$ for all $X \in \mathcal{X}(M)$, $\xi \in \mathcal{D}(M)$. It is seen that $J^{*2} = -1^*$ where 1^* denotes the identity map of $\mathcal{D}(M)$.

We shall denote by $\otimes^2TM$ ($\otimes^2T^*M$) the tensor product $TM \otimes TM$ ($T^*M \otimes T^*M$) of the tangent (cotangent) bundle of M by itself, as well as the total manifold of this bundle. The bundle will be called the *bitangent (bicotangent) tensor bundle* of M. By a *bicovariant (bicontravariant) tensor field* on M we shall mean a smooth section of this bundle. If Φ (F) is such a field, then by Φ_x (F_x) we denote the value of Φ (F) at $x \in M$. We say that Φ (F) is nondegenerate at a point x of M if the associated bilinear map

$$\Phi_x\colon T_xM \times T_xM \longrightarrow \mathbb{R} \ (F_x\colon T^*_xM \times T^*_xM \longrightarrow \mathbb{R})$$

is nondegenerate. If Φ (F) is nondegenerate at each point of M (an open subset U of M), we call it nondegenerate (on U). It is known that every bicovariant (bicontravariant) tensor field Φ (F) on M can be regarded as a $C^\infty(M)$-bilinear map

$$\Phi\colon \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow C^\infty(M) \ (F\colon \mathcal{D}(M) \times \mathcal{D}(M) \longrightarrow C^\infty(M)).$$

We shall denote by Φ^T (F^T) the transpose of Φ (F) defined by $\Phi^T(X,Y) = \Phi(Y,X)$ for $X,Y \in \mathcal{X}(M)$ ($F^T(\xi,\eta) = F(\eta,\xi)$ for $\xi,\eta \in \mathcal{D}(M)$). Call Φ (F) *symmetric* if $\Phi^T = \Phi$ ($F^T = F$) and *antisymmetric* if $\Phi^T = -\Phi$ ($F^T = -F$).

If $f\colon N \longrightarrow M$ is a smooth map of differentiable manifolds, we denote by

$$f^*\colon \otimes^k\mathcal{D}(M) \longrightarrow \otimes^k\mathcal{D}(N)$$

the pullback of the map f for covariant tensor fields of degree $k \geq 0$, where $\otimes^k\mathcal{D}(M)$ ($\otimes^k\mathcal{D}(N)$) denotes the $C^\infty(M)$-module ($C^\infty(N)$-module) of all k-covariant tensor fields on M (N). In particular, for $k = 0$ we have $f^*\colon C^\infty(M) \longrightarrow C^\infty(N)$ where $f^*\alpha = \alpha\circ f$. In turn, we denote by

$$f_*\colon \otimes^k TN \longrightarrow \otimes^k TM$$

the differential of the map f of degree $k \geq 1$, where $\otimes^k TN$ ($\otimes^k TM$) denotes the k-tangent tensor manifold of N (M). Throughout this paper we can restrict our attention to the case $k \leq 2$.

2. Conjugate pairs and J-connected pairs

Let M be a differentiable manifold and let Φ (F) be a bicovariant (bicontravariant) tensor field on M. We define the $C^\infty(M)$-linear maps $\Phi^a\colon \mathcal{X}(M) \longrightarrow \mathcal{D}(M)$ and $\Phi^b\colon \mathcal{X}(M) \longrightarrow \mathcal{D}(M)$ ($F^a\colon \mathcal{D}(M) \longrightarrow \mathcal{X}(M)$ and $F^b\colon \mathcal{D}(M) \longrightarrow \mathcal{X}(M)$) by

$$\Phi(X,Y) = \langle\Phi^aX,Y\rangle = \langle\Phi^bY,X\rangle \text{ for all } X,Y \in \mathcal{X}(M)$$

$$(F(\xi,\eta) = \langle\xi,F^b\eta\rangle = \langle\eta,F^a\xi\rangle \text{ for all } \xi,\eta \in \mathcal{D}(M)).$$

This means that Φ^a and Φ^b (F^a and F^b) are mutually adjoint operations with respect to the differential pairing of M. Thus, we can write $\Phi^{a*} = \Phi^b$ and $\Phi^{b*} = \Phi^a$ ($F^{a*} = F^b$ and $F^{b*} = F^a$). Clearly, $\Phi^aX = i_X\Phi = \Phi(X,\cdot)$ and $\Phi^bY = i_Y\Phi^T = \Phi(\cdot,Y)$ ($F^a\xi = i_\xi F = F(\xi,\cdot)$ and $F^b\eta = i_\eta F^T = F(\cdot,\eta)$). Moreover, note that

$$\Phi^{x*} = \Phi^{Tx} \text{ and } F^{x*} = F^{Tx} \text{ for } x = a,b.$$

By an easy verification we get

2.1. Proposition. *Let* Φ (F) *be a bicovariant* (*bicontravariant*) *tensor field on* M. *Then the following conditions are equivalent*:

(a) $F^b \circ \Phi^a = 1$;

(b) $F^a \circ \Phi^b = 1$;

(c) $\Phi^b \circ F^a = 1^*$;

(d) $\Phi^a \circ F^b = 1^*$;

(e) $\langle \Phi^a X, F^a \xi \rangle = \langle \xi, X \rangle$ *for all* $X \in \mathcal{X}(M)$, $\xi \in \mathcal{D}(M)$;

(f) $\langle \Phi^b X, F^b \xi \rangle = \langle \xi, X \rangle$ *for all* $X \in \mathcal{X}(M)$, $\xi \in \mathcal{D}(M)$. ∎

By a *pair of 2-tensor fields* on M we shall mean a pair (Φ,F) where Φ (F) is a bicovariant (bicontravariant) tensor field on M. Such a pair is called *conjugate* if it satisfies at least one from the equivalent conditions of Proposition 2.1. Note that if (Φ,F) is a conjugate pair of 2-tensor fields on M, then both Φ and F are nondegenerate. Moreover, the maps Φ^a and Φ^b (F^a and F^b) are $C^\infty(M)$-linear isomorphisms from $\mathcal{X}(M)$ onto $\mathcal{D}(M)$ (from $\mathcal{D}(M)$ onto $\mathcal{X}(M)$). Clearly, Proposition 2.1 implies

2.2. Corollary. *Let* (Φ,F) *be a conjugate pair of 2-tensor fields on* M.

(1) (Φ^T,F^T) *is conjugate.*

(2) *If* $\alpha,\beta \in C^\infty(M)$ *and* $\alpha\beta = 1$, *then* $(\alpha\Phi,\beta F)$ *is conjugate. In particular, so is* $(-\Phi,-F)$. ∎

Let Φ (F) be a bicovariant (bicontravariant) tensor field on M. By definition Φ (F) is *symmetric* if $\Phi^T = \Phi$ ($F^T = F$), or equivalently, $\Phi^a = \Phi^b$ ($F^a = F^b$) and *antisymmetric* if $\Phi^T = -\Phi$ ($F^T = -F$), or equivalently, $\Phi^a = -\Phi^b$ ($F^a = -F^b$). From Proposition 2.1 we obviously get

2.3. Corollary. *Let* (Φ,F) *be a conjugate pair of 2-tensor fields on* M. *Then* Φ *is symmetric* (*antisymmetric*) *if and only if so is* F. ∎

If Φ (F) is a nondegenerate bicovariant (bicontravariant) tensor field on M, then there is a unique nondegenerate bicontravariant (bicovariant) tensor field $\Phi^\sim$ ($^\sim F$) on M such that $(\Phi,\Phi^\sim)$ $((^\sim F,F))$ is a conjugate pair of 2-tensor fields on M. Namely, $\Phi^\sim$ ($^\sim F$) is defined by

$$\Phi^\sim(\xi,\eta) = \langle \xi,(\Phi^a)^{-1}\eta \rangle = \langle \eta,(\Phi^b)^{-1}\xi \rangle \text{ for } \xi,\eta \in \mathcal{D}(M)$$

$$(^\sim F(X,Y) = \langle (F^b)^{-1}X,Y \rangle = \langle (F^a)^{-1}Y,X \rangle \text{ for } X,Y \in \mathcal{X}(M)).$$

Clearly, the assignments $\Phi \mapsto \Phi^\sim$ ($F \mapsto {}^\sim F$) are mutually inverse, that is, $^\sim(\Phi^\sim) = \Phi$ $((^\sim F)^\sim = F)$. Thus we have

2.4. Corollary. *The assignment* $\Phi \mapsto \Phi^{\sim}$ $(F \mapsto {}^{\sim}F)$ *defines a one-to-one correspondence between nondegenerate bicovariant (bicontravariant) tensor fields on* M *and nondegenerate bicontravariant (bicovariant) tensor fields on* M. *Moreover, these assignments are mutually inverse.* ■

Let F be an arbitrary bicontravariant tensor field defined on a differentiable manifold M. For any $\alpha \in C^{\infty}(M)$, we define the *canonical left (right)* F-*vector field* of α to be the vector field $F^a_\alpha = F^a(d\alpha)$ $(F^b_\alpha = F^b(d\alpha))$. It is seen that the assignment $F^a_\bullet: \alpha \mapsto F^a_\alpha$ $(F^b_\bullet: \alpha \mapsto F^b_\alpha)$ is an $\mathcal{X}(M)$-valued derivation of the algebra $C^{\infty}(M)$ called the *canonical left (right)* F-*derivation* of $C^{\infty}(M)$. Clearly, we have $F^a_\alpha(\beta) = \langle(d\alpha)\otimes(d\beta),F\rangle$ and $F^b_\alpha(\beta) = \langle(d\alpha)\otimes(d\beta),F^T\rangle$, and so, $F^a_\alpha(\beta) = F^b_\beta(\alpha)$ for all $\alpha,\beta \in C^{\infty}(M)$.

Note that if (Φ,F) is a conjugate pair of 2-tensor fields on M, then

$$\Phi(X,F^a_\beta) = \langle\Phi^a X,F^a(d\beta)\rangle = \langle d\beta,X\rangle \text{ for } X \in \mathcal{X}(M),\ \beta \in C^{\infty}(M).$$

Similarly, we have

$$\Phi(F^b_\alpha,Y) = \langle\Phi^b Y,F^b(d\alpha)\rangle = \langle d\alpha,Y\rangle \text{ for } Y \in \mathcal{X}(M),\ \alpha \in C^{\infty}(M).$$

By putting $X = F^a_\alpha$ and $Y = F^b_\beta$ in the above equalities we obtain

$$\begin{aligned}\Phi(F^a_\alpha,F^a_\beta) &= \langle d\beta,F^a_\alpha\rangle = F(d\alpha,d\beta);\\ \Phi(F^b_\alpha,F^b_\beta) &= \langle d\alpha,F^b_\beta\rangle = F(d\alpha,d\beta).\end{aligned} \tag{2.1}$$

From the last equalities and Proposition 2.1 we get

2.5. Proposition. *Let* (Φ,F) *be a conjugate pair of* 2-*tensor fields on* M.

(1) $\Phi(F^a\xi,F^a\eta) = \Phi(F^b\xi,F^b\eta) = F(\xi,\eta)$ *for all* $\xi,\eta \in \mathcal{D}(M)$, *or equivalently,* $(F^a\otimes F^a)\Phi = (F^b\otimes F^b)\Phi = F$.

(2) $F(\Phi^a X,\Phi^a Y) = F(\Phi^b X,\Phi^b Y) = \Phi(X,Y)$ *for all* $X,Y \in \mathcal{X}(M)$, *or equivalently,* $(\Phi^a\otimes\Phi^a)F = (\Phi^b\otimes\Phi^b)F = \Phi$. ■

By an easy computation we get

2.6. Proposition. *Let* (Φ,F) *be a conjugate pair of* 2-*tensor fields defined on* M. *Let* $\Phi|U = \Sigma_{ij}\phi_{ij}(dx^i)\otimes(dx^j)$ $(F|U = \Sigma_{ij}f^{ij}(\partial/\partial x^i)\otimes(\partial/\partial x^j))$ *be the expression of* Φ (F) *in a local chart* $(U;x^1,\dots,x^n)$ *on* M. *Then the matrices* $\{\phi_{ij}\}$ *and* $\{f^{ij}\}^T$, *or equivalently,* $\{\phi_{ij}\}^T$ *and* $\{f^{ij}\}$ *are mutually inverse, i.e.*

$$\Sigma_k f^{ki}\phi_{kj} = \Sigma_k f^{ik}\phi_{jk} = \langle dx^i,\partial/\partial x^j\rangle \text{ for } i,j = 1,2,\dots,n.$$

Moreover, the following equalities hold:

(1) $\Phi^a X = \Sigma_{ij}\lambda^i\phi_{ij}dx^j$, $\Phi^b X = \Sigma_{ij}\lambda^j\phi_{ij}dx^i$;

(2) $F^a\xi = \Sigma_{ij}\alpha_i f^{ij}(\partial/\partial x^j)$, $F^b\xi = \Sigma_{ij}\alpha_j f^{ij}(\partial/\partial x^i)$;

(3) $F^a_\alpha = \Sigma_{ij}(\partial\alpha/\partial x^i)f^{ij}(\partial/\partial x^j)$, $F^b_\alpha = \Sigma_{ij}(\partial\alpha/\partial x^j)f^{ij}(\partial/\partial x^i)$;

where $X = \Sigma_i \lambda^i(\partial/\partial x^i)$, $\xi = \Sigma_i \alpha_i dx^i$ *and* $\alpha, \alpha_i, \lambda^i \in C^\infty(U)$ *for* $i = 1,\dots,n$. ∎

If M is a differentiable manifold, then by a *biderivation* of the algebra $C^\infty(M)$ we shall mean an ℝ-bilinear map $(\cdot,\cdot)$ on $C^\infty(M)$ satisfying the following conditions:

$$(\gamma, \alpha\beta) = \beta(\gamma,\alpha) + \alpha(\gamma,\beta);$$
$$(\alpha\beta, \gamma) = \beta(\alpha,\gamma) + \alpha(\beta,\gamma).$$

One can see that for every biderivation $(\cdot,\cdot)$ of $C^\infty(M)$ there is a unique bicontravariant tensor field F on M such that $(\alpha,\beta) = F(d\alpha,d\beta)$ for $\alpha,\beta \in C^\infty(M)$, called the *representative tensor field* of $(\cdot,\cdot)$. Conversely, if F is such a field, then the formula

$$(\alpha,\beta)_F = F(d\alpha,d\beta)$$

defines a biderivation of $C^\infty(M)$ called the F-*biderivation*. In the sequel the term an F-*biderivation* of $C^\infty(M)$ will always mean a biderivation of $C^\infty(M)$ with its representative tensor field F. Note that an F-biderivation of $C^\infty(M)$ is symmetric (antisymmetric) if and only if so is F, i.e. $F^T = F$ ($F^T = -F$). Let now Φ be a nondegenerate bicovariant tensor field on M. In this case the $\Phi^\sim$-biderivation will be also called the Φ-*biderivation* and denoted by $(\cdot,\cdot)_\Phi$. From (2.1) it follows that

$$(\alpha,\beta)_\Phi = \Phi^\sim(d\alpha,d\beta) = \Phi((\Phi^\sim)^a_\alpha,(\Phi^\sim)^a_\beta) = \Phi((\Phi^\sim)^b_\alpha,(\Phi^\sim)^b_\beta) \text{ for } \alpha,\beta \in C^\infty(M).$$

If (M,ω) is a symplectic manifold, then the ω-biderivation of $C^\infty(M)$ is called the *Poisson bracket* on $C^\infty(M)$ and denoted by $\{\alpha,\beta\} = (\alpha,\beta)_\omega$. It is known that this bracket is antisymmetric and it satisfies additionally the Jacobi identity, which means that the linear space $C^\infty(M)$ is a Lie algebra under multiplication given by the Poisson bracket. Furthermore, for any $\alpha \in C^\infty(M)$ the vector field $X_\alpha = (\omega^\sim)^b_\alpha$ is called the *Hamiltonian vector field* of α. These vector fields play an essential role in theoretical mechanics [1]. Let us set $X^-_\alpha = (\omega^\sim)^a_\alpha = -X_\alpha$ for $\alpha \in C^\infty(M)$. It is seen that

$$\omega(X_\alpha,X_\beta) = \omega(X^-_\alpha,X^-_\beta) = \{\alpha,\beta\} \text{ for } \alpha,\beta \in C^\infty(M).$$

Moreover, the Jacobi identity implies that the assignment $\alpha \mapsto X_\alpha$ ($\alpha \mapsto X^-_\alpha$) defines a Lie algebra antihomomorphism (homomorphism) from $(C^\infty(M),\{\cdot,\cdot\})$ to $(\mathcal{X}(M),[\cdot.\cdot])$.

Let now (M,g) be a pseudo-Riemannian manifold. By the *pseudo-Riemannian bracket* on $C^\infty(M)$ we shall mean the g-biderivation $(\alpha,\beta) = (\alpha,\beta)_g$. Clearly, this bracket is symmetric, that is, $(\alpha,\beta) = (\beta,\alpha)$ for $\alpha,\beta \in C^\infty(M)$. If $\alpha \in C^\infty(M)$, we define the *pseudo-Riemannian vector field* of α to be $R_\alpha = (g^\sim)^a_\alpha = (g^\sim)^b_\alpha$. It is seen that

$$g(R_\alpha,R_\beta) = (\alpha,\beta) \text{ for } \alpha,\beta \in C^\infty(M).$$

We shall regard the algebra $C^\infty(M)$ to be partially ordered by the relation $\geq$, where $\alpha \geq \beta$ means that $\alpha(x) \geq \beta(x)$ for all $x \in M$. We write $\alpha > \beta$ ($\alpha \gg \beta$) in case $\alpha \geq \beta$ and $\alpha \neq \beta$ ($\alpha(x) > \beta(x)$ for all $x \in M$). If in addition g is positive-defined, that is, (M,g) is a Riemannian manifold, we obviously have

$$(\alpha,\alpha) \geq 0 \text{ for all } \alpha \in C^\infty(M);$$
$$(\alpha,\alpha) > 0 \text{ if and only if } d\alpha \neq 0;$$
$$(\alpha,\alpha) \gg 0 \text{ if and only if } d_x\alpha \neq 0 \text{ for each } x \in M.$$

In this case the vector field R_α is called the *Riemannian vector field* of α, as well.

It is easy to verify

2.7. Proposition. *Let* (M,J) *be an almost complex manifold. Let* Φ (F) *be a bicovariant (bicontravariant) tensor field on* M. *Then the following conditions are equivalent*:

(a) $\Phi(JX,JY) = \Phi(X,Y)$ *for* $X,Y \in \mathcal{X}(M)$ ($F(J^*\xi,J^*\eta) = F(\xi,\eta)$ *for* $\xi,\eta \in \mathcal{D}(M)$);

(b) $(J^*\otimes J^*)\Phi = \Phi$ ($(J\otimes J)F = F$);

(c) $J^*\Phi^a J = \Phi^a$ ($JF^aJ^* = F^a$);

(d) $J^*\Phi^b J = \Phi^b$ ($JF^bJ^* = F^b$). ∎

Under the assumptions of Proposition 2.7, we say that Φ (F) is J-*invariant* if at least one from the equivalent conditions of this proposition is satisfied.

Let (M,J) be an almost complex manifold. If Φ and Ψ (F and G) are arbitrary bicovariant (bicontravariant) tensor fields on M, we set $\Phi^J = (J^*\otimes 1^*)\Phi$ and ${}^J\Psi = (1^*\otimes J^*)\Psi$ ($F^J = (J\otimes 1)F$ and ${}^JG = (1\otimes J)G$). Clearly, we have

$$\Phi^J(X,Y) = \Phi(JX,Y) \text{ and } {}^J\Psi(X,Y) = \Psi(X,JY) \text{ for } X,Y \in \mathcal{X}(M)$$
$$(F^J(\xi,\eta) = F(J^*\xi,\eta) \text{ and } {}^JG(\xi,\eta) = G(\xi,J^*\eta) \text{ for } \xi,\eta \in \mathcal{D}(M)).$$

By an easy verification we get

2.8. Proposition. *Let* (M,J) *be an almost complex manifold.*

(1) *The assignments* $\Phi \mapsto \Phi^J$ and $\Psi \mapsto {}^J\Psi$ ($F \mapsto F^J$ and $G \mapsto {}^JG$) *are mutually inverse bijections for* J-*invariant bicovariant (bicontravariant) tensor fields on* M.

(2) *These assignments define a one-to-one correspondence between antisymmetric and symmetric* J-*invariant bicovariant (bicontravariant) tensor fields on* M. ∎

It is easy to prove

2.9. Proposition. *Let* (M,J) *be an almost complex manifold. Let* Φ *and* Ψ (F *and* G) *be J-invariant bicovariant* (*bicontravariant*) *tensor fields on* M. *Then the following conditions are equivalent*:

(a) $\Phi^J = \Psi$ ($F^J = G$);
(b) ${}^J\Psi = \Phi$ (${}^JG = F$);
(c) $\Phi^a J = \Psi^a$ ($F^a J^* = G^a$);
(d) $\Phi^a = J^*\Psi^a$ ($F^a = JG^a$);
(e) $J^*\Phi^b = \Psi^b$ ($JF^b = G^b$);
(f) $\Phi^b = \Psi^b J$ ($F^b = G^b J^*$). ∎

Let (M,J) be an almost complex manifold. By a *J-connected bicovariant* (*bicontravariant*) *pair* on M we shall mean a pair (Φ,Ψ) ((F,G)) satisfying at least one from the equivalent conditions of Proposition 2.9. If in addition Φ (F) is antisymmetric and Ψ (G) is symmetric, we say that (Φ,Ψ) ((F,G)) is a *J-connected bicovariant* (*bicontravariant*) *superpair* on M. A J-connected bicovariant (bicontravariant) pair (Φ,Ψ) ((F,G)) on M is said to be *nondegenerate* if both Φ and Ψ (F and G) are nondegenerate. It turns out that for an arbitrary almost complex manifold (M,J) there are nondegenerate J-connected bicovariant (bicontravariant) superpairs on M. Indeed, we can choose a J-invariant metric tensor g on M, and note that $({}^Jg,g)$ $(({}^J(\tilde{g}),\tilde{g}))$ is such a superpair.

It is seen that Propositions 2.1 and 2.9 imply

2.10. Corollary. *Let* (M,J) *be an almost complex manifold. Let* (Φ,F) *and* (Ψ,G) *be conjugate pairs of 2-tensor fields on* M. *Then* (Φ,Ψ) *is a bicovariant J-connected pair* (*superpair*) *if and only if* (F,G) *is a bicontravariant one. Moreover, if* (Φ,Ψ) *is a J-connected pair on* M, *then* $F^a_\alpha = JG^a_\alpha$ *and* $JF^b_\beta = G^b_\beta$ *for all* $\alpha,\beta \in C^\infty(M)$. ∎

This corollary immediately implies

2.11. Corollary. *Let* (M,J) *be an almost complex manifold. Let* (ω,g) *be a J-connected bicovariant superpair on* M *where* ω (g) *is a symplectic form* (*pseudo-Riemannian metric*) *on* M. *Then*

$$X^-_\alpha = JR_\alpha \text{ and } JX_\beta = R_\beta \text{ for all } \alpha,\beta \in C^\infty(M). \;\blacksquare$$

Let (M,J) be a complex manifold, i.e. an almost complex manifold satisfying the following integrable condition:

$$[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y] = 0 \text{ for all } X,Y \in \mathcal{X}(M).$$

A nondegenerate J-connected bicovariant superpair (ω,g) on M is said to be *integrable* if so is ω, i.e. ω is a symplectic form ($d\omega = 0$). Such a superpair defines a nondegenerate complex bicovariant tensor field on M which is regarded to be a complex analogue of the symplectic structure ω (see [10] for details). Moreover, if (ω,g) is an integrable nondegenerate J-connected bicovariant superpair on M, then g can be regarded as a symmetric analogue of ω. It turns out that the above analogues can be considered more generally for complex manifolds equipped with J-invariant Poisson structure.

3. Regular and strictly regular points

Let M be a differentiable manifold. Let $\mathcal{E}$ be an assignment which sends each point $x \in M$ to a vector subspace $\mathcal{E}_x$ of T_xM. Let us set

$$\Gamma(\mathcal{E}) = \{X \in \mathcal{X}(M): X_x \in \mathcal{E}_x \ \forall\ x \in M\}$$

and note that $\Gamma(\mathcal{E})$ is a $C^\infty(M)$-submodule of $\mathcal{X}(M)$. We say that $\mathcal{E}$ is a *distribution* on M if for any $p \in M$ and $v \in \mathcal{E}_p$ there is a vector field $X \in \Gamma(\mathcal{E})$ passing through v, i.e. $X_p = v$. If $\mathcal{E}$ is a distribution on M, we define the dimension function $\delta_{\mathcal{E}}$ on M by $\delta_{\mathcal{E}}(x) = \dim \mathcal{E}_x$. One can see that this function is lower semicontinuous. We say that $p \in M$ is a *regular point* of $\mathcal{E}$ if $\delta_{\mathcal{E}}$ is locally constant at p. Denote by reg $\mathcal{E}$ the set of all regular points of $\mathcal{E}$. By an easy verification we get

3.1. Lemma. *If* $\mathcal{E}$ *is a distribution on* M, *then* reg $\mathcal{E}$ *is an open and dense subset of* M. ∎

Let U be an open subset of M. A distribution $\mathcal{E}$ on M is called *regular* (on U) in case reg $\mathcal{E}$ = M (U ⊆ reg $\mathcal{E}$). We say that $\mathcal{E}$ is of *dimension k* (on U) in case $\delta_{\mathcal{E}}(x) = k$ for each $x \in M$ ($x \in U$), and write $\dim \mathcal{E} = k$ ($\dim \mathcal{E}|U = k$). Clearly, if M is connected, then every regular distribution on M is of constant dimension.

Let F be a bicontravariant tensor field on M. We associate with F the *left* (*right*) *distribution* $\mathcal{E}^a(F)$ ($\mathcal{E}^b(F)$) of F on M defined by

$$\mathcal{E}^a(F)_x = F^a(T^*_xM) \quad (\mathcal{E}^b(F)_x = F^b(T^*_xM)).$$

Note that in any local chart $(U;x^1,\dots,x^n)$ on M the field F has the expression

$$F|U = \Sigma_{ij} f^{ij}(\partial/\partial x^i)\otimes(\partial/\partial x^j).$$

This implies that equalities (2) and (3) of Proposition 2.6 are satisfied for F too. For each $x \in U$, we define the rank $\rho_F(x)$ of F at x to be the rank of the matrix $\{f^{ij}(x)\}$. Clearly, the definition of $\rho_F(x)$ does not depend from a chosen local chart containing x. Thus we can regard that the rank-dimension function ρ_F: $x \mapsto \rho_F(x)$ is well-defined on M. Moreover, it is seen that for each $x \in M$ we have

$$\rho_F(x) = \dim F^a(T_x^*M) = \dim F^b(T_x^*M),$$

which implies that $\rho_F = \delta_{\mathcal{E}^a(F)} = \delta_{\mathcal{E}^b(F)}$. Therefore ρ_F is a lower semicontinuous function on M because so are $\delta_{\mathcal{E}^a(F)}$ and $\delta_{\mathcal{E}^b(F)}$. By a *regular point* of F we shall mean a point $p \in M$ such that the function ρ_F is locally constant at p. The set of all such points will be denoted by reg F. Since $\rho_F = \delta_{\mathcal{E}^a(F)} = \delta_{\mathcal{E}^b(F)}$, we get

$$\operatorname{reg} F = \operatorname{reg} \mathcal{E}^a(F) = \operatorname{reg} \mathcal{E}^b(F).$$

This and Lemma 3.1 immediately imply

3.2. Corollary. *If F is a bicontravariant tensor field on M, then* reg F *is an open and dense subset of* M. ∎

Let U be an open subset of M. We say that F is *regular* (on U) in case reg F = M ($U \subseteq \operatorname{reg} F$), which means that F is locally of constant rank (on U). It is seen that F is regular (on U) if and only if so are $\mathcal{E}^a(F)$ and $\mathcal{E}^b(F)$ simultaneously. We write rank F = k (rank $F|U = k$) if $\rho_F(x) = k$ for each $x \in M$ ($x \in U$). Let us set

$$\operatorname{reg}^{\dagger} F = \{x \in M\colon \rho_F(x) = \dim M\}$$

and note that $\operatorname{reg}^{\dagger} F$ is an open subset of reg F. Clearly, F is nondegenerate if and only if $\operatorname{reg}^{\dagger} F = M$. We say that F is quasi-nondegenerate if $\operatorname{reg}^{\dagger} F$ is a dense subset of M. In this case we obviously have $\operatorname{reg}^{\dagger} F = \operatorname{reg} F$.

Let U be an open subset of M. If $\mathcal{S}$ is a family of vector fields on M, we adopt the following notation:

$$\mathcal{S}|U = \{X|U\colon X \in \mathcal{S}\}.$$

By a *module* of *vector fields* on U we shall mean a $C^\infty(U)$-submodule of $\mathfrak{X}(U)$. If $\mathfrak{N}$ is such a module and if $\mathcal{S}$ is a system of generators of $\mathfrak{N}$, we write

$$\mathfrak{N} = \operatorname{lin}_{C^\infty(U)} \mathcal{S}.$$

Let $\mathfrak{M}$ be a module of vector fields on M. For an open subset U of M we denote by $\mathfrak{M}(U)$ the $C^\infty(U)$-module generated by the family $\mathfrak{M}|U$, i.e. $\mathfrak{M}(U) = \operatorname{lin}_{C^\infty(U)} (\mathfrak{M}|U)$. Moreover, by $\mathfrak{M}_{loc}(U)$ we denote the *localization* of $\mathfrak{M}$ to U. This means that $\mathfrak{M}_{loc}(U)$ is a module of vector fields on U consisting of all $X \in \mathfrak{X}(U)$ satisfying the following condition:

for each $p \in U$ *there are an open neighbourhood* V *of* p *in* U *and a vector field* $Y \in \mathfrak{M}$ *such that* $X|V = Y|V$.

We say that $\mathfrak{M}$ is *local* on U in case $\mathfrak{M}_{loc}(U) = \mathfrak{M}(U)$. If $\mathfrak{M}$ is local on M, i.e. $\mathfrak{M}_{loc}(M) = \mathfrak{M}$, we shortly say that it is *local*. Clearly, if $\mathfrak{M}$ is local, then it is local on each open subset of M. A family $\mathcal{S} \subseteq \mathfrak{M}$ is said to be a *system* of *local* *generators* of $\mathfrak{M}$ if for any $X \in \mathfrak{M}$ and $p \in M$ there is an open neighbourhood U of p in M such that $X|U \in \mathrm{lin}_{C^\infty(U)} (\mathcal{S}|U)$. We say that a family $\mathcal{S} \subseteq \mathfrak{M}$ is a *system* of *locally neighbourhood generators* of $\mathfrak{M}$ if for each $p \in M$ one can find an open neighbourhood U of p in M such that $\mathfrak{M}(U) = \mathrm{lin}_{C^\infty(U)} (\mathcal{S}|U)$. Clearly, every system of locally neighbourhood generators of $\mathfrak{M}$ is a system of local generators of $\mathfrak{M}$. If U is an open subset of M, then by a *system* of *generators* (*base*) of $\mathfrak{M}$ on U we shall mean a system of generators (base) of the module $\mathfrak{M}(U)$.

On principle we deal with local modules of vector fields on M. For example, if $\mathcal{E}$ is a distribution on M, then the module $\Gamma(\mathcal{E})$ is clearly local. By applying a locally finite smooth partition of unity we get

3.3. Lemma. *Let* $\mathfrak{M}$ *be a module of vector fields on* M.

(1) *A finite system of local generators of* $\mathfrak{M}$ *is a system of generators of* $\mathfrak{M}$.

(2) *If* $\mathfrak{M}$ *is finitely generated, then it is local.* ∎

Let $\mathfrak{M}$ be a module of vector fields on M and let $p \in M$. We call $\mathfrak{M}$ *pseudoregular* at p in case there is an open neighbourhood U of p in M such that the module $\mathfrak{M}(U)$ is finitely generated. Clearly, the set of all pseudoregular points of $\mathfrak{M}$ is an open subset of M. We say that $\mathfrak{M}$ is *pseudoregular* if it is pseudoregular at each point of M.

Let $\mathfrak{M}$ be a module of vector fields on M and let U be an open subset of M. A base $X^1,\ldots,X^k$ of $\mathfrak{M}(U)$ is said to be *regular* if the vectors $X^1_x,\ldots,X^k_x$ are linearly independent for each $x \in U$. By a *regular point* of $\mathfrak{M}$ we shall mean a point $p \in M$ such that there is a regular base of $\mathfrak{M}(U)$ for some open neighbourhood U of p in M. Clearly, the set $\mathrm{reg}\,\mathfrak{M}$ of all regular points of $\mathfrak{M}$ is an open subset of M. We say that $\mathfrak{M}$ is *regular* on U in case $U \subseteq \mathrm{reg}\,\mathfrak{M}$. If $\mathrm{reg}\,\mathfrak{M} = M$, then $\mathfrak{M}$ is called *regular.*

It turns out that a free module of vector fields on M need not be regular in general. For example, let $X = x(\partial/\partial x)$ be a vector field on $\mathbb{R}$ and consider the module $\mathfrak{M} = \mathrm{lin}_{C^\infty(\mathbb{R})} \{X\}$. Clearly, $\mathfrak{M}$ is free. But for any open neighbourhood U of 0 in $\mathbb{R}$ every base of $\mathfrak{M}(U)$ consists of one nonzero vector field on U which vanishes at 0, and so, $\mathfrak{M}$ is not regular at 0.

Let $\mathfrak{M}$ be a module of vector fields on M. By the *distribution* of $\mathfrak{M}$ we shall mean a distribution $\mathcal{E}(\mathfrak{M})$ on M defined by

$$\mathcal{E}(\mathfrak{M})_x = \{X_x \colon X \in \mathfrak{M}\}.$$

It is seen that $\mathfrak{M}$ is a submodule of $\Gamma(\mathcal{E}(\mathfrak{M}))$. For example, consider the vector field $X = x^2(\partial/\partial x)$ on $\mathbb{R}$. Let us set $\mathfrak{M} = \mathrm{lin}_{C^\infty(\mathbb{R})} \{X\}$ and $\mathcal{E} = \mathcal{E}(\mathfrak{M})$. We have $\mathcal{E}_x = T_x\mathbb{R}$ for $x \neq 0$ and $\mathcal{E}_0 = 0$. Moreover, note that the vector field $x(\partial/\partial x)$

belongs to $\Gamma(\mathcal{E})$ but it does not belong to $\mathfrak{M}$.

Let $\mathcal{E}$ be a distribution on M. A family $\mathcal{S} \subseteq \Gamma(\mathcal{E})$ is said to be a *system* of *smooth generators* of $\mathcal{E}$ if it is a system of generators of the module $\Gamma(\mathcal{E})$. Similarly, we define a *system* of *local* (*locally neighbourhood*) *smooth generators* of $\mathcal{E}$, as well as a *system* of *smooth generators* (*base*) of $\mathcal{E}$ on an *open subset* of M. We say that $\mathcal{E}$ is *pseudoregular* at a point $p \in M$ if so is the module $\Gamma(\mathcal{E})$ at p. By a *pseudoregular distribution* on M we shall mean a distribution which is pseudoregular at each point of M. It is easy to verify

3.4. Lemma. *Let $\mathfrak{M}$ be a module of vector fields on M and let $\mathcal{E} = \mathcal{E}(\mathfrak{M})$.*

(1) *If $\mathcal{S}$ is a system of generators of $\mathfrak{M}$, then for each $p \in \operatorname{reg} \mathcal{E}$ there are an open neighbourhood U of p in M and vector fields $X^1,\dots,X^k \in \mathcal{S}$ such that $\{X^1|U,\dots,X^k|U\}$ is a regular base of $\mathfrak{M}$ on U. In particular,*

$$\operatorname{reg} \mathfrak{M} = \operatorname{reg} \mathcal{E},$$

and so, reg $\mathfrak{M}$ *is an open and dense subset of* M.

(2) *For each $p \in \operatorname{reg} \mathfrak{M}$ there is an open neighbourhood U of p in M such that $\mathfrak{M}(U) = \Gamma(\mathcal{E}|U)$. In particular, if $\mathfrak{M}$ is regular, then a family $\mathcal{S} \subseteq \mathfrak{M}$ is a system of local (locally neighbourhood) generators of $\mathfrak{M}$ if and only if it is a system of local (locally neighbourhood) smooth generators of $\mathcal{E}$.* ∎

This lemma immediately implies

3.5. Corollary. *Let $\mathfrak{M}$ be a local module of vector fields on* M *and let $\mathcal{E} = \mathcal{E}(\mathfrak{M})$. If* U *is an open subset of* M *and $U \subseteq \operatorname{reg} \mathfrak{M}$, then $\mathfrak{M}(U) = \Gamma(\mathcal{E}|U)$. In particular, if $\mathfrak{M}$ is regular, then $\mathfrak{M} = \Gamma(\mathcal{E})$.* ∎

Let F be a bicontravariant tensor field on M. As we know F^a and F^b are $C^\infty(M)$-linear maps form $\mathcal{D}(M)$ to $\mathcal{X}(M)$. This implies that $\mathfrak{M}^a(F) = F^a(\mathcal{D}(M))$ and $\mathfrak{M}^b(F) = F^b(\mathcal{D}(M))$ are modules of vector fields on M called the *canonical left* and *right modules* of F, respectively. Let us set $\{F^a_\alpha\} = \{F^a_\alpha\colon \alpha \in C^\infty(M)\}$ and $\{F^b_\alpha\} = \{F^b_\alpha\colon \alpha \in C^\infty(M)\}$.

3.6. Theorem. *Let* F *be a bicontravariant tensor field on* M.

(1) *The family $\{F^a_\alpha\}$ ($\{F^b_\alpha\}$) is a system of locally neighbourhood generators of $\mathfrak{M}^a(F)$ ($\mathfrak{M}^b(F)$).*

(2) *The module $\mathfrak{M}^a(F)$ ($\mathfrak{M}^b(F)$) is pseudoregular and local.*

(3) *If in addition* M *is connected, then $\mathfrak{M}^a(F)$ ($\mathfrak{M}^b(F)$) is a finitely generated module by $\{F^a_\alpha\}$ ($\{F^b_\alpha\}$).*

Proof. By reason of symmetry, it suffices to show this theorem for $\{F^a_\alpha\}$ and $\mathfrak{M}^a(F)$. For convenience, we adopt the notations $\mathfrak{M}^a = \mathfrak{M}^a(F)$ and $\{F^a_\alpha|V\} = \{F^a_\alpha\}|V$.

(1). Let us take $p \in M$ and consider a local chart $(U;x^1,\dots,x^n)$ on M containing p. Since F^a is a local operator, we can accept $F^a(\xi|U) = (F|U)^a(\xi|U) = (F^a\xi)|U$ for $\xi \in \mathcal{D}(M)$, which implies $F^a(\mathcal{D}(M)|U) = \mathfrak{M}^a|U$. Clearly, we have $x^1,\dots,x^n \in C^\infty(U)$, and so, there are an open neighbourhood V of p in U and functions $\alpha^1,\dots,\alpha^n \in C^\infty(M)$ such that $\alpha^i|V = x^i|V$ for $i = 1,\dots,n$. Thus $d\alpha^i|V = dx^i|V$ and since the differentials $dx^1,\dots,dx^n$ form a base of the module $\mathcal{D}(U)$, it follows that the differentials $d\alpha^1|V,\dots,d\alpha^n|V$ form a base of the module $\mathcal{D}(V)$. Finally, it is seen that $F^a(d\alpha^i|V) = F^a_{\alpha^i}|V$ for $i = 1,\dots,n$ and $F^a(\mathcal{D}(V)) = \mathfrak{M}^a(V)$, whence

$$\mathfrak{M}^a(V) = \mathrm{lin}_{C^\infty(V)}\,(F^a_{\alpha^1}|V,\dots,F^a_{\alpha^n}|V).$$

Consequently, $\{F^a_\alpha\}$ is a system of locally neighbourhood generators of $\mathfrak{M}^a$.

(2). Observe first that from the prove of Statement (1) it follows that the module $\mathfrak{M}^a(V)$ is finitely generated, which means that $\mathfrak{M}^a$ is pseudoregular at p. Since p can be an arbitrary point of M, we conclude that $\mathfrak{M}^a$ is pseudoregular.

To prove that the module $\mathfrak{M}^a$ is local consider a vector field $X \in \mathfrak{M}^a_{loc}(M)$. This means that for any point $p \in M$ there is an open neighbourhood U of p in M such that $X|U \in \mathfrak{M}^a(U)$. Since $F^a(\mathcal{D}(U)) = \mathfrak{M}^a(U)$, there is $\omega_U \in \mathcal{D}(U)$ such that $F^a(\omega_U) = X|U$. In turn, there are an open neighbourhood V of p in U and a differential form $\xi^V \in \mathcal{D}(M)$ such that $\xi^V|V = \omega_U|V$, which implies $F^a(\xi^V|V) = X|V$. Thus there is an open covering $\mathcal{V}$ of M and a family $\{\xi^V\colon V \in \mathcal{V}\}$ such that

$$F^a(\xi^V|V) = X|V \text{ for each } V \in \mathcal{V}. \tag{3.1}$$

Let $\{\lambda_s\colon s \in S\}$ be a locally finite smooth partition of unity subordinated to $\mathcal{V}$. For each $s \in S$ we can choose $V(s) \in \mathcal{V}$ such that $\operatorname{supp}\lambda_s \subseteq V(s)$. Define the differential 1-form ξ on M by

$$\xi = \Sigma_{s\in S}\,\lambda_s\xi^{V(s)}.$$

Clearly, we have $F^a\xi = \Sigma_{s\in S}\,\lambda_s F^a\xi^{V(s)} \in \mathfrak{M}^a$. It suffices to show that $F^a\xi = X$, or equivalently, that for each $p \in M$ there is an open neighbourhood U of p in M such that

$$(F^a\xi)|U = X|U.$$

Indeed, there is an open neighbourhood W of p in M such that the set

$$S_p = \{s \in S\colon W \cap \operatorname{supp}\lambda_s \neq \emptyset\}$$

is finite. Moreover, the set

$$S'_p = \{s \in S\colon p \in \operatorname{supp}\lambda_s\}$$

is a finite subset of S_p. Note that $p \in \operatorname{supp} \lambda_s \subseteq V(s)$ for $s \in S'_p$, and so, $\bigcap \{V(s): s \in S'_p\}$ is an open neighbourhood of p in M. Consider the set

$$U = (W \cap \bigcap \{V(s): s \in S'_p\}) \setminus \bigcup \{\operatorname{supp} \lambda_s: s \in S_p \setminus S'_p\}$$

and note that U is an open neighbourhood of p in M such that the following conditions hold:

(3.2) $U \subseteq V(s)$ *for each* $s \in S'_p$;

(3.3) $U \cap \operatorname{supp} \lambda_s = \emptyset$ *for each* $s \in S \setminus S'_p$.

From conditions (3.1) and (3.2) it follows that $F^a(\xi^{V(s)}|U) = X|U$ for each $s \in S'_p$. This and condition (3.3) imply

$$(F^a\xi)|U = \Sigma_{s\in S} (\lambda_s|U)F^a(\xi^{V(s)}|U) = \Sigma_{s\in S'_p} (\lambda_s|U)(X|U) = X|U,$$

which completes the proof of statement (2).

(3). If M is connected, then by the Whitney theorem there exists a smooth embedding $\varphi = (\varphi^1,\dots,\varphi^m): M \longrightarrow \mathbb{R}^m$. It follows that every global differential 1-form ξ on M is a pullback of such a form on $\mathbb{R}^m$ via φ, i.e.

$$\xi = \varphi^*(\Sigma_{i=1}^m \alpha_i dx^i) = \Sigma_{i=1}^m (\varphi^*\alpha_i)d\varphi^i$$

where $\alpha_i \in C^\infty(\mathbb{R}^m)$. Thus, $\mathcal{D}(M)$ is finitely generated by the differentials $d\varphi^1,\dots,d\varphi^m$. This implies that the module $\mathfrak{M}^a = F^a(\mathcal{D}(M))$ is also finitely generated by the vector fields $F^a(d\varphi^1),\dots,F^a(d\varphi^m)$. ∎

For this theorem one can remark that if M is connected, then the statement (3) and Lemma 3.3 imply the statement (2).

From Theorem 3.6 and Lemma 3.4 we get

3.7. Corollary. *Let* F *be a bicontravariant tensor field on* M. *For any* $p \in \operatorname{reg} F$ *there are an open neighbourhood* U *of* p *in* M *and functions* $\alpha^1,\dots,\alpha^k \in C^\infty(M)$ $(\beta^1,\dots,\beta^k \in C^\infty(M))$ *where* $k = \rho_F(p)$ *such that the vector fields*

$$F^a(d\alpha^1)|U,\dots,F^a(d\alpha^k)|U \quad (F^b(d\beta^1)|U,\dots,F^b(d\beta^k)|U)$$

form a regular base of $\mathfrak{M}^a(F)$ $(\mathfrak{M}^b(F))$ *on* U. *In other words, there is a local chart* $(U;x^1,\dots,x^n)$ $((U;y^1,\dots,y^n))$ *on* M *containing* p *such that the vector fields*

$$F^a(dx^1),\dots,F^a(dx^k) \quad (F^b(dy^1),\dots,F^b(dy^k))$$

form a regular base of $\mathfrak{M}^a(F)$ $(\mathfrak{M}^b(F))$ *on* U. ∎

Let U be an open subset of M. A module $\mathfrak{M}$ of vector fields on M is said to be *involutive* on U in case the module $\mathfrak{M}(U)$ is involutive, i.e. if $X,Y \in \mathfrak{M}(U)$ involves $[X,Y] \in \mathfrak{M}(U)$. If $\mathfrak{M}$ is involutive on M, we shortly say that it is *involutive*. If $p \in M$, we call $\mathfrak{M}$ *locally involutive* at p in case there is an open neighbourhood U of p in M such that $\mathfrak{M}(U)$ is involutive. We say that $\mathfrak{M}$ is *involutive* at a *point* $p \in M$ in case for any vector fields $X,Y \in \mathfrak{M}$ there is an open neighbourhood U of p in M such that $[X|U,Y|U] \in \mathfrak{M}(U)$. It is easy to verify the following lemmas.

3.8. Lemma. *Let $\mathfrak{M}$ be a pseudoregular module of vector fields on* M. *Then $\mathfrak{M}$ is locally involutive at a point $p \in M$ if and only if it is involutive at* p. ■

3.9. Lemma. *Let $\mathfrak{M}$ be a local module of vector fields on* M. *Then $\mathfrak{M}$ is involutive on an open subset* U *of* M *if and only if it is involutive at each point of* U. ■

By applying Theorem 3.6 and Lemma 3.9 we get

3.10. Proposition. *Let* F *be a bicontravariant tensor field on* M *and let* U *be an open subset of* M. *Let us set $\mathfrak{M}^a = \mathfrak{M}^a(F)$ ($\mathfrak{M}^b = \mathfrak{M}^b(F)$). Then the following conditions are equivalent:*

(a) *the module $\mathfrak{M}^a(U)$ ($\mathfrak{M}^b(U)$) is involutive;*

(b) $[F^a_\alpha|U,F^a_\beta|U] \in \mathfrak{M}^a(U)$ $([F^b_\alpha|U,F^b_\beta|U] \in \mathfrak{M}^b(U))$ *for all* $\alpha,\beta \in C^\infty(M)$;

(c) *there is an open covering $\mathcal{V}$ of* U *such that $\mathfrak{M}^a(V)$ ($\mathfrak{M}^b(V)$) is involutive for each $V \in \mathcal{V}$;*

(d) $\mathfrak{M}^a$ ($\mathfrak{M}^b$) *is involutive at each point of* U. ■

This proposition immediately implies

3.11. Corollary. *Under the assumptions of Proposition* 3.10, *the module $\mathfrak{M}^a$ ($\mathfrak{M}^b$) is involutive on* U *if and only if for each point $p \in U$ there is a local chart $(V;x^1,\dots,x^n)$ on* U *containing* p *such that*

$$[F^a(dx^i),F^a(dx^j)] \in \mathfrak{M}^a(V) \ ([F^b(dx^i),F^b(dx^j)] \in \mathfrak{M}^b(V)) \ \textit{for } i,j = 1,2,\dots,n. \ ■$$

Let F be a bicontravariant tensor field on M. If $p \in M$, we call F *left* (*right*) *locally involutive* at p in case there is an open neighbourhood U of p in M such that at least one from the equivalent conditions of Proposition 3.10 is satisfied. We say that F is *left* (*right*) *involutive* at p in case the module $\mathfrak{M}^a(F)$ ($\mathfrak{M}^b(F)$) is involutive at p. Note that from statement (2) of Theorem 3.6 and from Lemma 3.8 we get

3.12. Corollary. *Let* F *be a bicontravariant tensor field on* M *and let* $p \in M$. *Then* F *is left* (*right*) *locally involutive at* p *if and only if it is left* (*right*) *involutive at* p. ■

Let F be a bicontravariant tensor field on M and let U be an open subset of M.

We say that F is *left* (*right*) *involutive* on U in case it is left (right) involutive at each point of U, or equivalently, it satisfies at least one from the equivalent conditions of Proposition 3.10. If F is such on M, we shortly say that it is *left* (*right*) *involutive*. We call F *involutive* (on U) in case it is left and right involutive (on U) simultaneously. If $\mathfrak{M}^a(F) = \mathfrak{M}^b(F)$, we say that F is *balanced*. Recall that by definition F is symmetric (antisymmetric) if and only if $F^a = F^b$ ($F^a = -F^b$). Moreover, F is nondegenerate if and only if $\mathfrak{M}^a(F) = \mathfrak{M}^b(F) = \mathfrak{X}(M)$. Hence we conclude that if F is symmetric, antisymmetric and nondegenerate, respectively, then it is balanced. The following example shows that F can be left involutive but not right involutive, and conversely. Clearly, every such field is not balanced.

3.13. Example. For any $p = (x,y,z) \in \mathbb{R}^3$ we consider the matrix

$$\begin{Bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x \end{Bmatrix}$$

which defines the bicontravariant tensor field F on $\mathbb{R}^3$ by

$$F = (\partial/\partial x)\otimes(\partial/\partial x) + (\partial/\partial x)\otimes(\partial/\partial y) + (\partial/\partial y)\otimes(\partial/\partial z) + x(\partial/\partial z)\otimes(\partial/\partial z)$$

Clearly, F is of constant rank 2. We have

$$F^a(dx) = \partial/\partial x + \partial/\partial y,\ F^a(dy) = \partial/\partial z,\ F^a(dz) = x(\partial/\partial z);$$

$$F^b(dx) = F^b(dy) = \partial/\partial x,\ F^b(dz) = \partial/\partial y + x(\partial/\partial z).$$

Thus, $[F^a(dx),F^a(dy)] = [F^a(dy),F^a(dz)] = 0$ and $[F^a(dx),F^a(dz)] = \partial/\partial z$, which implies that the module $\mathfrak{M}^a(F)$ is involutive (Corollary 3.11), and so, F is left involutive. On the other hand, note that $[F^b(dx),F^b(dz)] = \partial/\partial z \notin \mathfrak{M}^b(F)$, which means that the module $\mathfrak{M}^b(F)$ is not involutive, and so, F is not right involutive. Obviously, the bicontravariant tensor field F^T is right involutive but not left involutive. ■

Let F be a bicontravariant tensor field on M. We say that $p \in M$ is a *left* (*right*) *strictly regular point* of F if $p \in \text{reg}\,F$ and if F is left (right) involutive at p, or equivalently, if it is left (right) locally involutive at p (Corollary 3.12). Denote by $\text{reg}^a\,F$ ($\text{reg}^b\,F$) the set of all left (right) strictly regular points of F. Clearly, by definition $\text{reg}^a\,F$ ($\text{reg}^b\,F$) is an open subset of M. By a *strictly regular point* of F we shall mean a point $p \in M$ which is a left and right strictly regular point of F simultaneously. It is seen that

$$\text{reg}^*\,F = \text{reg}^a\,F \cap \text{reg}^b\,F$$

is the set of all strictly regular points of F. Note that if F is the bicontravariant tensor field on $\mathbb{R}^3$ from Example 3.13, then $\mathrm{reg}^a F = \mathrm{reg}\, F = \mathbb{R}^3$ and $\mathrm{reg}^* F = \mathrm{reg}^b F = \emptyset$. Clearly, we get

3.14. Proposition. *If* F *is left (right) involutive, then* $\mathrm{reg}^a F = \mathrm{reg}\, F$ ($\mathrm{reg}^b F = \mathrm{reg}\, F$). *In particular, if* F *is involutive, then* $\mathrm{reg}^* F = \mathrm{reg}^a F = \mathrm{reg}^b F = \mathrm{reg}\, F$. ∎

If F is a bicontravariant tensor field on M, we define the *left (right) polar algebra* of F to be the subalgebra $\mathcal{P}^a(F)$ ($\mathcal{P}^b(F)$) of $C^\infty(M)$ given by the following equalities:

$$\mathcal{P}^a(F) = \ker F^b_\bullet = \{\alpha \in C^\infty(M) \colon (\alpha,\beta)_F = 0 \ \forall\ \beta \in C^\infty(M)\}$$
$$= \{\alpha \in C^\infty(M) \colon X(\alpha) = 0 \ \forall\ X \in \mathfrak{M}^b(F)\}$$
$$(\mathcal{P}^b(F) = \ker F^a_\bullet = \{\beta \in C^\infty(M) \colon (\alpha,\beta)_F = 0 \ \forall\ \alpha \in C^\infty(M)\}$$
$$= \{\beta \in C^\infty(M) \colon X(\beta) = 0 \ \forall\ X \in \mathfrak{M}^a(F)\}).$$

One can see that the algebra $\mathcal{P}^a(F)$ ($\mathcal{P}^b(F)$) is *closed under composition with real smooth functions*, which means that if $\alpha^1,\dots,\alpha^m \in \mathcal{P}^a(F)$ ($\alpha^1,\dots,\alpha^m \in \mathcal{P}^b(F)$) and $\varphi \in C^\infty(\mathbb{R}^m)$, then $\varphi\circ(\alpha^1,\dots,\alpha^m) \in \mathcal{P}^a(F)$ ($\varphi\circ(\alpha^1,\dots,\alpha^m) \in \mathcal{P}^b(F)$). Moreover, it is seen that this algebra is *local*, that is, if $\alpha \in C^\infty(M)$ and if for each $p \in M$ there are an open neighbourhood U of p in M and a function $\beta \in \mathcal{P}^a(F)$ ($\beta \in \mathcal{P}^b(F)$) such that $\alpha|U = \beta|U$, then $\alpha \in \mathcal{P}^a(F)$ ($\alpha \in \mathcal{P}^b(F)$).

Let M be a differentiable manifold. We say that functions $\alpha^1,\dots,\alpha^k \in C^\infty(M)$ are *differentially independent* at a point p of M in case the differentials $d_p\alpha^1,\dots,d_p\alpha^k$ are linearly independent in T^*_pM. If U is an open subset of M and if these functions are differentially independent at each point of M (U), we call $\alpha^1,\dots,\alpha^k$ *differentially independent* (on U). Clearly, if $\alpha^1,\dots,\alpha^k$ are differentially independent at p, then there is an open neighbourhood U of p in M such that these functions are differentially independent on U.

If $\mathcal{A}$ is a subalgebra of $C^\infty(M)$ and if $x \in M$, we denote by $d_x\mathcal{A}$ the linear subspace of T^*_xM consisting of all differentials $d_x\alpha$ for $\alpha \in \mathcal{A}$. Define the *left (right) polar-dimension function* of F to be the function $\delta^a_F \colon M \longrightarrow \mathbb{Z}^+$ ($\delta^b_F \colon M \longrightarrow \mathbb{Z}^+$) given by

$$\delta^a_F(x) = \dim d_x\mathcal{P}^a(F) \ (\delta^b_F(x) = \dim d_x\mathcal{P}^b(F)).$$

It is seen that $k = \delta^a_F(x)$ ($k = \delta^b_F(x)$) is the maximal number of differentially independent at x functions from $\mathcal{P}^a(F)$ ($\mathcal{P}^b(F)$). From definition it follows that δ^a_F (δ^b_F) is a lower semicontinuous function. Moreover, note that these functions are *locally defined* by F, which means that for any open subset U of M we have $\delta^a_F|U = \delta^a_{F|U}$ and $\delta^b_F|U = \delta^b_{F|U}$. Further, define the *left (right) total-dimension function* of F to be the function $\tau^a_F \colon M \longrightarrow \mathbb{Z}^+$ ($\tau^b_F \colon M \longrightarrow \mathbb{Z}^+$) given by

$$\tau_F^a(x) \doteq \delta_F^a(x) + \rho_F(x) \ (\tau_F^b(x) = \delta_F^b(x) + \rho_F(x)).$$

Since both the functions δ_F^a and ρ_F (δ_F^b and ρ_F) are lower semicontinuous, so is τ_F^a (τ_F^b). Clearly, the functions τ_F^a and τ_F^b as well as the function ρ_F are locally defined by F, too. Moreover, note that

(3.4) $$\rho_F(x) \le \tau_F^a(x) \le \dim M \ (\rho_F(x) \le \tau_F^b(x) \le \dim M)$$

for each $x \in M$.

Let U be an open subset of a differentiable manifold M and let $p \in M$. A distribution $\mathcal{E}$ on M is said to be *involutive* on U, *locally involutive* at p and *involutive* at p, respectively, if so is the module $\Gamma(\mathcal{E})$.

3.15. Theorem. *If* F *is a bicontravariant tensor field on* M, *then*

$$\mathrm{reg}^a\, F = \{x \in M: \tau_F^b(x) = \dim M\} \ (\mathrm{reg}^b\, F = \{x \in M: \tau_F^a(x) = \dim M\}).$$

Proof. By reason of symmetry, it suffices to show that

$$\mathrm{reg}^a\, F = \{x \in M: \tau_F^b(x) = \dim M\}.$$

Let us set $\mathfrak{M}^a = \mathfrak{M}^a(F)$ and $\mathcal{E}^a = \mathcal{E}^a(F)$. Let $p \in \mathrm{reg}^a\, F$ and $k = \mathrm{rank}\, F_p = \rho_F(p)$. Since p is a regular point of F and F is left involutive at p, we can find an open neighbourhood U of p in M such that $\mathfrak{M}^a(U)$ is regular (Corollary 3.7) and involutive (Proposition 3.10). Clearly, $\dim \mathfrak{M}^a(U) = k$ and $\mathfrak{M}^a(U) = \Gamma(\mathcal{E}^a|U)$ by Corollary 3.5. Thus $\mathcal{E}^a$ is a regular involutive distribution on U and $\dim \mathcal{E}^a|U = k$. From the Frobenius theorem it follows that there is a k-dimensional foliation $\mathcal{F}$ on U such that every leaf of $\mathcal{F}$ is a maximal integrable manifold of $\mathcal{E}^a|U$ (see [4]). More precisely, we can find a local foliation chart $(V,\varphi) \in \mathcal{F}$ such that $p \in V \subseteq U$ and $\varphi(V) = W' \times W'' \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ where $n = \dim M$ and W' and W'' are open disks in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively. Let $\pi''\colon W' \times W'' \longrightarrow W''$ be the projection onto the second factor and let $\varphi'' = \pi'' \circ \varphi$. Since every plague $\mathcal{P}_c = \varphi^{-1}(W' \times \{c\})$, $c \in W''$ is a k-dimensional integrable manifold of $\mathcal{E}^a|V$, it follows that each vector field from $\mathfrak{M}^a(V) = \Gamma(\mathcal{E}^a|V)$ is tangent to $\mathcal{P}_c$. Thus for each $\alpha \in C^\infty(W'')$ the function $\varphi''^*\alpha \in C^\infty(V)$ is constant on every such plague, and so, $X(\varphi''^*\alpha) = 0$ for each $X \in \mathfrak{M}^a(V) = \mathfrak{M}^a(F|V)$. In other words, we have $\varphi''^*(C^\infty(W'')) \subseteq \mathcal{P}^b(F|V)$, which implies that for each $x \in V$ the linear pullback map $\varphi_x''^*$ transforms $T^*_{\varphi''(x)}W''$ to $T^*_x\mathcal{P}^b(F|V)$. Since the smooth map $\varphi''\colon V \longrightarrow W''$ is surjective, it follows that $\varphi_x''^*$ is a linear monomorphism, and so,

$$\delta^b_{F|V}(x) = \dim T^*_x\mathcal{P}^b(F|V) \ge \dim T^*_{\varphi''(x)}W'' = \dim W'' = n-k.$$

Since δ_F^b is locally defined by F, we get $\delta_F^b(x) = \delta^b_{F|V}(x) \ge n-k$ for $x \in V$. On the other hand, from (3.4) we get $\delta^b_F(x) \le n - \rho_F(x) = n-k$ for $x \in U$, and so, $\delta^b_F(x) = n-k$ for $x \in V$. Thus $\tau^b_F(x) = \delta^b_F(x) + \rho_F(x) = n$ for $x \in V$, which proves the inclusion $\operatorname{reg}^a F \subseteq \{x \in M: \tau^b_F(x) = \dim M\}$.

To prove the converse inclusion, suppose that p is a point of M such that $\tau^b_F(p) = \dim M = n$. Since both the functions δ^b_F and ρ_F are lower semicontinuous on M, there is an open neighbourhood U of p in M such that $\delta^b_F(x) \ge \delta^b_F(p)$ and $\rho_F(x) \ge \rho_F(p)$ for $x \in U$. But $\tau^b_F(x) = \delta^b_F(x) + \rho_F(x) \le n$ for each $x \in M$, and so, δ^b_F and ρ_F are constant functions on U, i.e. $\delta^b_F(x) = \delta^b_F(p) = n-k$ and $\rho_F(x) = \rho_F(p) = k$ for $x \in U$. Consequently, $p \in \operatorname{reg} F$ and there are functions $\alpha^1,\ldots,\alpha^m \in \mathcal{P}^b(F)$ ($m = n-k$) which are differentially independent at p. This implies that they are such on some open neighbourhood U′ of p in U. On the other hand, $X(\alpha^i) = 0$ for $X \in \mathfrak{M}^a$ and since $p \in \operatorname{reg} F = \operatorname{reg} \mathfrak{M}^a$ and rank $F|U = k$, there is a local chart $(V;x^1,\ldots,x^n)$ on M such that $p \in V \subseteq U'$, the vector fields $X^1 = F^a(dx^1),\ldots,X^k = F^a(dx^k)$ form a regular base of $\mathfrak{M}^a(V)$ and $X^j(\alpha^i|V) = 0$ for $i = 1,\ldots,m$ and $j = 1,\ldots,k$. It remains to prove that $\mathfrak{M}^a(V)$ is involutive. Indeed, if $X,Y \in \mathfrak{M}^a(V)$, then for $Z = [X,Y]$ we have

$$Z_x(\alpha^i) = 0 \ (i = 1,\ldots,m) \text{ for } x \in V. \tag{3.5}$$

Note that for each $x \in V$ the set

$$L_x = \{v \in T_xM: v(\alpha^i) = 0 \ (i = 1,\ldots,m)\}$$

is a linear subspace of T_xM and $\dim L_x = n-m = k$ because the differentials $d_x\alpha^1,\ldots,d_x\alpha^m$ are linearly independent. Since for each $x \in V$ the vectors $X^1_x,\ldots,X^k_x$ form a base of the linear space $\mathcal{E}^a_x$ and $\mathcal{E}^a_x \subseteq L_x$, it follows that $\mathcal{E}^a_x = L_x$. Hence and from (3.5) we get $Z \in \Gamma(\mathcal{E}^a|V) = \mathfrak{M}^a(V)$ (Corollary 3.5), which means that $\mathfrak{M}^a(V)$ is involutive. This completes the proof that $p \in \operatorname{reg}^a F$. ∎

By applying this theorem we get

3.16. Corollary. *Let* F *be a bicontravariant tensor field on* M *and let* $F|U = \sum_{ij} f^{ij}(\partial/\partial x^i)\otimes(\partial/\partial x^j)$ *be the expression of* F *in a local chart* $(U;x^1,\ldots,x^n)$ *on* M. *If* $p \in U$ *is a left (right) strictly regular point of* F *and* $k = \delta^b_F(p)$ ($k = \delta^a_F(p)$), *then there exists an open neighbourhood* V *of* p *in* U *such that the system*

$$\sum_{j=1}^{n} f^{ij}(x)(\partial\alpha/\partial x^j) = 0 \ (i = 1,\ldots,n)$$

$$\left(\sum_{i=1}^{n} f^{ij}(x)(\partial\alpha/\partial x^i) = 0 \ (j = 1,\ldots,n)\right)$$

of differential equations on V *has maximally k differentially independent solu-*

tions $\alpha^1,\dots,\alpha^k \in C^\infty(\dot{V})$. *Moreover, if* $\alpha \in C^\infty(V)$ *is an arbitrary solution of this system, then there is a function* $\varphi \in C^\infty(\mathbb{R}^k)$ *such that* $\alpha = \varphi\circ(\alpha^1,\dots,\alpha^k)$. ■

Let F be a bicontravariant tensor field on M. We say that $p \in M$ is a *left* (*right*) *coregular point* of F in case $\delta^a_F(p) = 0$ ($\delta^b_F(p) = 0$). Since δ^a_F (δ^b_F) is a lower semicontinuous nonnegative function, it follows that the set

$$\mathrm{coreg}^a\, F = \{x \in M\colon \delta^a_F(x) = 0\} \ (\mathrm{coreg}^b\, F = \{x \in M\colon \delta^b_F(x) = 0\})$$

is closed in M. We call F *left* (*right*) *coregular* in case $\mathrm{coreg}^a\, F = M$ ($\mathrm{coreg}^b\, F = M$). If F is left and right coregular simultaneously, we say shortly that it is *coregular*.

An F-biderivation $(\cdot,\cdot)_F$ of $C^\infty(M)$ is said to be *left* (*right*) *nondegenerate* in case $\mathcal{P}^a(F) = 0$ ($\mathcal{P}^b(F) = 0$). If $(\cdot,\cdot)_F$ is left and right nondegenerate simultaneously, we call shortly it *nondegenerate*.

It is seen that if F is an arbitrary bicontravariant tensor field F on M, then F is left (right) coregular if and only if $(\cdot,\cdot)_F$ is left (right) nondegenerate. Hence and from Proposition 3.14 and Theorem 3.15 we get

3.17. Proposition. *Let* F *be an involutive bicontravariant tensor field on* M. *Then the following conditions are equivalent*:

(a) F *is left* (*right*) *coregular*;

(b) $(\cdot,\cdot)_F$ *is left* (*right*) *nondegenerate*;

(c) F *is coregular*;

(d) $(\cdot,\cdot)_F$ *is nondegenerate*;

(e) F *is quasi-nondegenerate*. ■

Observe that the assumption of this proposition is essential. Indeed, the bicontravariant tensor field F on $\mathbb{R}^3$ from Example 3.13 is left involutive but not right involutive. Moreover, F is not quasi-nondegenerate and not left coregular but it is right coregular.

4. The category of vector-derived manifolds

As we know, any bicontravariant tensor field F on a differentiable manifold M determines the canonical left derivation $F^a_\bullet$ of the algebra $C^\infty(M)$. Conversely, if Δ is an $\mathcal{X}(M)$-valued derivation of $C^\infty(M)$, then there is a unique bicontravariant tensor field F on M such that $F^a_\bullet = \Delta$. Thus, the assignment $F \mapsto F^a_\bullet$ defines a one-to-one correspondence between bicontravariant tensor fields on M and $\mathcal{X}(M)$-valued derivations of $C^\infty(M)$. Let $\mathfrak{I}\colon \mathcal{X}(M) \longrightarrow \mathcal{D}(M)$ be a fixed $C^\infty(M)$-linear isomorphism. Note that any such isomorphism can be of the form Φ^b for a unique bicovariant tensor field Φ on M, namely defined by $\Phi(X,Y) = \langle \mathfrak{I}Y,X\rangle$ for $X,Y \in \mathcal{X}(M)$.

If F is a bicontravariant tensor field on M, then the composition $\mathfrak{I}\circ F^{a}_{\bullet}$ defines a $\mathcal{D}(M)$-valued derivation of $C^{\infty}(M)$. In particular, if (Φ,F) is a conjugate pair of 2-tensor fields on M and if $\mathfrak{I} = \Phi^{b}$, then this composition equals the exterior derivative d on $C^{\infty}(M)$. Note that every $\mathcal{D}(M)$-valued derivation of $C^{\infty}(M)$ is of the form $\mathfrak{I}\circ F^{a}_{\bullet}$ for some bicontravariant tensor field F on M.

H.-J. Kim [5] called a pair (M,D) *derived manifold* in case M is a differentiable manifold and D is a $\mathcal{D}(M)$-valued derivation of $C^{\infty}(M)$ such that $D\circ D = 0$ (integrable condition) where D: $\Lambda^{k}\mathcal{D}(M) \longrightarrow \Lambda^{k+1}\mathcal{D}(M)$ $(1 \le k \le \dim M)$ is uniquely defined by the conditions

$$Dd + dD = 0 \text{ and } D(\xi \wedge \eta) = D(\xi) \wedge \eta + (-1)^{k}\xi \wedge D(\eta)$$

for $\xi \in \Lambda^{k}\mathcal{D}(M)$ and $\eta \in \Lambda^{l}\mathcal{D}(M)$. More precisely, his definition of derived manifold is formulated in the case of complex-valued forms on M but it has a real-valued analogue as above. By analogy, we define a *vector-derived manifold*, shortly called an *$\mathcal{X}$-derived manifold*, to be a pair (M,F) where M is a differentiable manifold and F is a bicontravariant tensor field on M called an *$\mathcal{X}$-derived structure* on M, as well. Note that if $\mathfrak{I}$: $\mathcal{X}(M) \longrightarrow \mathcal{D}(M)$ is an arbitrary $C^{\infty}(M)$-isomorphism, then the $\mathcal{D}(M)$-valued derivation $D = \mathfrak{I}\circ F^{a}_{\bullet}$ of $C^{\infty}(M)$ need not satisfy $D\circ D = 0$, i.e. (M,D) need not be a derived manifold. In this sense our definition of vector-derived ($\mathcal{X}$-derived) manifold is more general than that of derived manifold (we do not require any integrable condition for F).

Let f: N $\longrightarrow$ M be a smooth map of differentiable manifolds. Denote by $\mathcal{X}_{f}(M)$ the $C^{\infty}(N)$-module of all f-vector fields on N tangent to M. We have the $C^{\infty}(N)$-linear map f_{*}: $\mathcal{X}(N) \longrightarrow \mathcal{X}_{f}(M)$ defined by $(f_{*}X)(\alpha) = X(f^{*}\alpha)$ for $\alpha \in C^{\infty}(M)$. Similarly, we define the map f_{*} for bicontravariant tensor fields. If K is an arbitrary tensor field on M, then by $f^{\#}K = K\circ f$ we denote the pullback of K via f. In particular, for any $X \in \mathcal{X}(M)$ we have $f^{\#}X \in \mathcal{X}_{f}(M)$.

4.1. Theorem. *Let* (M,F) *and* (N,G) *be $\mathcal{X}$-derived manifolds. If f*: N $\longrightarrow$ M *is a smooth map, then the following conditions are equivalent*:

(a) $G(f^{*}\xi,f^{*}\eta) = f^{*}F(\xi,\eta)$ *for all* $\xi,\eta \in \mathcal{D}(M)$;

(b) $(f^{*}\alpha,f^{*}\beta)_{G} = f^{*}(\alpha,\beta)_{F}$ *for all* $\alpha,\beta \in C^{\infty}(M)$;

(c) $f_{*}\circ G^{a}_{f^{*}\alpha} = f^{\#}F^{a}_{\alpha}$ *for each* $\alpha \in C^{\infty}(M)$;

(d) $f_{*}\circ G^{b}_{f^{*}\beta} = f^{\#}F^{b}_{\beta}$ *for each* $\beta \in C^{\infty}(M)$;

(e) $f_{*}\circ G^{a}\circ f^{*} = f^{\#}F^{a}$;

(f) $f_{*}\circ G^{b}\circ f^{*} = f^{\#}F^{b}$;

(g) $f_{*}\circ G = f^{\#}F$.

Proof. Our proof will run according to the scheme (a)⇒(b)⇒(c)⇒(e)⇒(g)⇒(a). This is sufficient because the complemented scheme (a)⇒(b)⇒(d)⇒(f)⇒(a) of implications can be proved analogously.

The implication (a)⇒(b) is trivial.

(b)⇒(c). For any $\alpha,\beta \in C^{\infty}(M)$ we have

$$(f_*\circ G^a_{f^*\alpha})(\beta) = G^a_{f^*\alpha}(f^*\beta) = (f^*\alpha,f^*\beta)_G = f^*(\alpha,\beta)_F$$
$$= f^*F^a_\alpha(\beta) = (f^\# F^a_\alpha)(\beta).$$

(c)⇒(e). Since all homomorphisms of the left diagram of (4.1) are local operators, it suffices to show (e) for a system of local generators of the $C^{\infty}(M)$-module $\mathcal{D}(M)$, that is, for differential forms $d\alpha$ ($\alpha \in C^{\infty}(M)$). We have

$$(f_*\circ G^a\circ f^*)(d\alpha) = f_*\circ G^a(df^*\alpha) = f_*\circ G^a_{f^*\alpha} = f^\# F^a_\alpha = (f^\# F^a)(d\alpha).$$

(e)⇒(g). For any $\alpha,\beta \in C^{\infty}(M)$ we have

$$(f_*\circ G)(d\alpha,d\beta) = G(df^*\alpha,df^*\beta) = G^a_{f^*\alpha}(f^*\beta) = (f_*\circ G^a_{f^*\alpha})(\beta)$$
$$= ((f_*\circ G^a\circ f^*)(d\alpha))(\beta) = ((f^\# F^a)(d\alpha))(\beta) = f^*((F^a(d\alpha)(\beta))$$
$$= f^*F(d\alpha,d\beta) = (f^\# F)(d\alpha,d\beta).$$

Finally, note that the last implication (g)⇒(a) is trivial, which completes the proof. ∎

Note that conditions (c) and (d) of Theorem 4.1 mean that the following diagrams are respectively commutative:

$$\begin{array}{ccc} TN & \xrightarrow{f_*} & TM \\ {\scriptstyle G^a_{f^*\alpha}}\uparrow & & \uparrow{\scriptstyle F^a_\alpha} \\ N & \xrightarrow{f} & M \end{array} \qquad\qquad \begin{array}{ccc} TN & \xrightarrow{f_*} & TM \\ {\scriptstyle G^b_{f^*\beta}}\uparrow & & \uparrow{\scriptstyle F^b_\beta} \\ N & \xrightarrow{f} & M \end{array}$$

Similarly, conditions (e) and (f) of this theorem mean that the following ones are respectively commutative:

(4.1)
$$\begin{array}{ccc} \mathcal{D}(M) & \xrightarrow{f^\# F^a} & \mathfrak{X}_f(M) \\ {\scriptstyle f^*}\downarrow & & \uparrow{\scriptstyle f_*} \\ \mathcal{D}(N) & \xrightarrow{G^a} & \mathfrak{X}(N) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{D}(M) & \xrightarrow{f^\# F^b} & \mathfrak{X}_f(M) \\ {\scriptstyle f^*}\downarrow & & \uparrow{\scriptstyle f_*} \\ \mathcal{D}(N) & \xrightarrow{G^b} & \mathfrak{X}(N) \end{array}$$

Finally, note that the last condition of Theorem 4.1 means that the following diagram is commutative:

$$\begin{array}{ccc} \otimes^2 TN & \xrightarrow{f_*} & \otimes^2 TM \\ \uparrow G & & \uparrow F \\ N & \xrightarrow{f} & M \end{array}$$

By a *morphism* f: (N,G) $\longrightarrow$ (M,F) *of $\mathcal{X}$-derived manifolds*, or shortly, *$\mathcal{X}$-derived morphism* we shall mean a smooth map f: N $\longrightarrow$ M such that at least one from the equivalent conditions of Theorem 4.1 is satisfied. We get a category which is called the *category* of *vector-derived* ($\mathcal{X}$-*derived*) *manifolds*. An $\mathcal{X}$-derived manifold (M,F) is said to be *symmetric* (*antisymmetric*) if so is F. A *Poisson manifold* (M,F) is defined to be an antisymmetric $\mathcal{X}$-derived manifold such that the (alternating) Schouten-Nijenhuis bracket [F,F] equals 0 (see [2]). We have the *categories* of *symmetric* (*antisymmetric*) *$\mathcal{X}$-derived manifolds* and of *Poisson manifolds* which are defined to be full subcategories of the category of $\mathcal{X}$-derived manifolds. Clearly, the category of Poisson manifolds is a full subcategory of that of antisymmetric $\mathcal{X}$-derived manifolds.

We say that (N,G) is an *$\mathcal{X}$-derived submanifold* of (M,F) if N is a differentiable submanifold of M and the inclusion map i: N $\hookrightarrow$ M defines an $\mathcal{X}$-derived morphism from (N,G) to (M,F). In this case G is uniquely defined by F and we call it the *$\mathcal{X}$-derived structure* on N *induced* by F. Note that by condition (g) of Theorem 4.1 we can regard that $G = F|N$.

Let f: (N,G) $\longrightarrow$ (M,F) be a morphism of $\mathcal{X}$-derived manifolds. For a point p $\in$ N let us consider a local chart $(U;x^1,\dots,x^n)$ on N containing p and a local chart $(V;y^1,\dots,y^m)$ containing f(p). We can additionally assume that $f(U) \subseteq V$. Clearly, $f(x) = (\varphi^1(x),\dots,\varphi^m(x))$ for $x = (x^1,\dots,x^n) \in U$, where $\varphi^i \in C^\infty(U)$. As we know (Section 2) G and F have the following expressions in U and V respectively:

$$G_x = \sum_{i,j=1}^{n} g^{ij}(x)(\partial/\partial x^i)\otimes(\partial/\partial x^j);$$

$$F_y = \sum_{k,l=1}^{m} f^{kl}(y)(\partial/\partial y^k)\otimes(\partial/\partial y^l).$$

From condition (g) of Theorem 4.1 it follows that $f_* G_x = F_{f(x)}$ for $x \in U$. Hence and from the above expressions we get

$$f_* G_x = \sum_{i,j=1}^{n} g^{ij}(x) f_*(\partial/\partial x^i)\otimes f_*(\partial/\partial x^j)$$

$$= \sum_{i,j=1}^{n} g^{ij}(x)\Big(\sum_{k=1}^{m}(\partial\varphi^k/\partial x^i)(\partial/\partial y^k)\Big)\otimes\Big(\sum_{l=1}^{m}(\partial\varphi^l/\partial x^j)(\partial/\partial y^l)\Big)$$

$$= \sum_{k,l=1}^{m} \Big(\sum_{i,j=1}^{n} (\partial\varphi^k/\partial x^i) g^{ij}(x)(\partial\varphi^l/\partial x^j)\Big)(\partial/\partial y^k)\otimes(\partial/\partial y^l) = F_{f(x)},$$

which means that

$$f^{kl}(f(x)) = \sum_{i,j=1}^{n} (\partial\varphi^k/\partial x^i) g^{ij}(x)(\partial\varphi^l/\partial x^j) \text{ for } x \in U \; (k,l = 1,\dots,m).$$

The last equality implies

4.2. Corollary. *Let* (M,F) *be an* $\mathcal{X}$*-derived manifold.*

(1) *If* f: (N,G) $\longrightarrow$ (M,F) *is a morphism of* $\mathcal{X}$*-derived manifolds, then* $\rho_F(f(x)) \le \rho_G(x)$ *for each* $x \in N$.

(2) *If* (N,G) *is an* $\mathcal{X}$*-derived submanifold of* (M,F), *then* $\rho_G(x) = \rho_F(x)$ *for each* $x \in N$. *In particular,* reg F $\cap$ N $\subseteq$ reg G. ∎

Give attention that if (M,F) is an $\mathcal{X}$-derived manifold and N is an arbitrary differentiable submanifold of M, then the $\mathcal{X}$-derived structure on N induced by F need not exist in general (Corollary 4.3 and Example 4.8). If there exists such a structure, we say that N *admits* the *$\mathcal{X}$-derived structure induced* by F. Note that if (M,F) is a symmetric (antisymmetric) $\mathcal{X}$-derived manifold or if it is a Poisson manifold, then every $\mathcal{X}$-derived submanifold of (M,F) is again such a manifold.

Let M be a differentiable manifold. For any subset S of M we put $C^\infty(M;S) = \{\alpha \in C^\infty(M): \alpha|S = 0\}$. Clearly, $C^\infty(M;S)$ is an ideal of the algebra $C^\infty(M)$. A vector field $X \in \mathcal{X}(M)$ is called *tangent* to S if $X(C^\infty(M;S)) \subseteq C^\infty(M;S)$. Denote by $\mathcal{X}(M;S)$ the set of all vector fields from $\mathcal{X}(M)$ which are tangent to S. It is seen that $\mathcal{X}(M;S)$ is a submodule of $\mathcal{X}(M)$. Note that from condition (b) of Theorem 4.1, or equivalently, from conditions (c) and (d) of this theorem we get

4.3. Corollary. *If* f: (N,G) $\longrightarrow$ (M,F) *is a morphism of* $\mathcal{X}$*-derived manifolds, then* $F^a_\alpha, F^b_\alpha \in \mathcal{X}(M;f(N))$ *for all* $\alpha \in C^\infty(M)$. ∎

This corollary and statement (1) of Theorem 3.6 imply

4.4. Corollary. *If* f: (N,G) $\longrightarrow$ (M,F) *is a morphism of* $\mathcal{X}$*-derived manifolds, then* $\mathfrak{M}^a(F) \cup \mathfrak{M}^b(F) \subseteq \mathcal{X}(M;f(N))$. ∎

Let (M,F) be an $\mathcal{X}$-derived manifold and let f: N $\longrightarrow$ M be a smooth map of differentiable manifolds. Note that the above corollary gives a necessary condition for f to be a morphism f: (N,G) $\longrightarrow$ (M,F) of $\mathcal{X}$-derived manifolds for some $\mathcal{X}$-derived structure G on N. It is seen that for any open subset U of M the pair (U,F|U) is an $\mathcal{X}$-derived manifold called the $\mathcal{X}$-derived submanifold of (M,F) defined by U. Clearly, the inclusion map i: U $\hookrightarrow$ M defines an $\mathcal{X}$-derived morphism from (U,F|U) to (M,F). We have the following theorem which is partially converse to Corollary 4.3.

4.5. Theorem. *Let* (M,F) *be an* $\mathfrak{X}$*-derived manifold and let* N *be a differentiable submanifold of* M. *If* $F^a_\alpha, F^b_\beta \in \mathfrak{X}(M;N)$ *for all* $\alpha,\beta \in C^\infty(M)$, *then* N *admits the* $\mathfrak{X}$*-derived structure induced by* F.

Proof. Clearly, we can regard that $F^a_\alpha|N = \iota^\# F^a_\alpha$ and $F^b_\beta|N = \iota^\# F^b_\beta$ are vector fields on N for all $\alpha,\beta \in C^\infty(M)$ where ι: N $\hookrightarrow$ M denotes the inclusion map. Moreover, note that

$$(F^a_\alpha|N)(\beta|N) = F^a_\alpha(\beta)|N = F^b_\beta(\alpha)|N = (F^b_\beta|N)(\alpha|N). \tag{4.2}$$

For any $\lambda \in C^\infty(N)$ we have the vector fields G'_λ and G''_λ on N which are well-defined by

$$G'_\lambda(\beta|N) = (F^b_\beta|N)(\lambda) \text{ and } G''_\lambda(\alpha|N) = (F^a_\alpha|N)(\lambda).$$

Note that the assignments $G'_\bullet: \lambda \mapsto G'_\lambda$ and $G''_\bullet: \lambda \mapsto G''_\lambda$ are $\mathfrak{X}(N)$-valued derivations of the algebra $C^\infty(N)$. This implies that there are uniquely defined $C^\infty(N)$-linear maps G' and G'' from $\mathcal{D}(N)$ to $\mathfrak{X}(N)$ such that $G'(d\lambda) = G'_\lambda$ and $G''(d\lambda) = G''_\lambda$ for $\lambda \in C^\infty(N)$. We have thus defined the bicontravariant tensor fields $\overline{G}'$ and $\overline{G}''$ on N by

$$\overline{G}'(\xi,\eta) = \langle\eta,G'\xi\rangle \text{ and } \overline{G}''(\xi,\eta) = \langle\xi,G''\eta\rangle$$

for $\xi,\eta \in \mathcal{D}(N)$. One can see that $\overline{G}' = \overline{G}''$, or equivalently,

$$\overline{G}'(\xi,\eta) = \overline{G}''(\xi,\eta) \text{ for all } \xi,\eta \in \mathcal{D}(N).$$

To prove this equality it suffices to show that it is satisfied for a system of local generators of the $C^\infty(N)$-module $\mathcal{D}(N)$, that is, for $\xi = d(\alpha|N)$ and $\eta = d(\beta|N)$ where $\alpha,\beta \in C^\infty(M)$. Indeed, from (4.2) we get

$$\begin{aligned}\overline{G}'(d(\alpha|N),d(\beta|N)) &= G'_{\alpha|N}(\beta|N) = (F^b_\beta|N)(\alpha|N) = (F^a_\alpha|N)(\beta|N)\\ &= G''_{\beta|N}(\alpha|N) = \overline{G}''(d(\alpha|N),d(\beta|N)).\end{aligned}$$

Thus we can set $G = \overline{G}' = \overline{G}''$, which implies that $G^a = G'$ and $G^b = G''$.

To prove that (N,G) is an $\mathfrak{X}$-derived submanifold of (M,F), or equivalently, that the inclusion map ι defines an $\mathfrak{X}$-derived morphism from (N,G) to (M,F) it suffices to show, according to condition (b) of Theorem 4.1, that

$$(\alpha|N,\beta|N)_G = (\alpha,\beta)_F|N \text{ for } \alpha,\beta \in C^\infty(M).$$

Indeed, we have $(\alpha|N,\beta|N)_G = G^b_{\beta|N}(\alpha|N) = (F^a_\alpha|N)(\beta|N) = (\alpha,\beta)_F|N$. Consequently, G is the $\mathfrak{X}$-derived structure on N induced by F. ∎

This theorem and Corollaries 4.3 and 4.4 immediately imply

4.6. Corollary. *Let* (M,F) *be an* $\mathcal{X}$*-derived manifold. If* N *is a differentiable submanifold of* M, *then* N *admits the* $\mathcal{X}$*-derived structure induced by* F *if and only if*

$$F^a_\alpha, F^b_\beta \in \mathcal{X}(M;N) \text{ for all } \alpha,\beta \in C^\infty(M),$$

or equivalently, $\mathfrak{M}^a(F) \cup \mathfrak{M}^b(F) \subseteq \mathcal{X}(M;N)$. ∎

An $\mathcal{X}$-derived manifold (M,F) is said to be *nondegenerate* (*quasi-nondegenerate*) if so is the bicontravariant tensor field F. Note that Corollary 4.6 implies

4.7. Corollary. *Let* (M,F) *be a nondegenerate* $\mathcal{X}$*-derived manifold. If* N *is a differentiable submanifold of* M, *then* N *admits the* $\mathcal{X}$*-derived structure induced by* F *if and only if it is an open subset of* M. ∎

It turns out that if (M,F) is a quasi-nondegenerate $\mathcal{X}$-derived manifold, then the assertion of Corollary 4.7 can be satisfied (Example 4.8) or not satisfied (Example 4.9).

4.8. Example. Consider the $\mathcal{X}$-derived manifold $(\mathbb{R}^2,F)$ where

$$F = (x^2+y^2)(\partial/\partial x)\otimes(\partial/\partial x) + (\partial/\partial y)\otimes(\partial/\partial y).$$

It is seen that reg F = $\mathrm{reg}^\dagger$ F = $\mathbb{R}^2 \setminus \{(0,0)\}$, and so, F is quasi-nondegenerate. Let N be a differentiable submanifold of $\mathbb{R}^2$. If N is an open subset of $\mathbb{R}^2$, then it obviously admits the $\mathcal{X}$-derived structure induced by F.

Conversely, suppose that N admits the $\mathcal{X}$-derived structure G induced by F, i.e. (N,G) is an $\mathcal{X}$-derived submanifold of $(\mathbb{R}^2,F)$. If dim N = 2, then N is obviously an open subset of $\mathbb{R}^2$. By Corollary 4.2 we have

$$2 = \rho_F(x) = \rho_G(x) \leq \dim N \text{ for } x \in N \setminus \{(0,0)\},$$

which leads to a contradiction in case dim N = 1. Finally, if dim N = 0, then the above-mentioned corollary implies

$$1 \leq \rho_F(x) = \rho_G(x) = 0 \text{ for } x \in N,$$

a contradiction. ∎

4.9. Example. Consider the $\mathcal{X}$-derived manifold $(\mathbb{R}^2,F)$ where

$$F = x(\partial/\partial x)\otimes(\partial/\partial x) + y(\partial/\partial y)\otimes(\partial/\partial y).$$

It is seen that F is quasi-nondegenerate and reg F = $\mathrm{reg}^\dagger$ F = $\mathbb{R}^2 \setminus \mathbb{K}$ where $\mathbb{K} = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$. Let us set $M = \mathbb{R} \times \{0\}$ and $N = \{0\} \times \mathbb{R}$ and consider the $\mathcal{X}$-derived manifolds (M,F′) and (N,F″) where $F' = x(\partial/\partial x)\otimes(\partial/\partial x)$ and $F'' = y(\partial/\partial y)\otimes(\partial/\partial y)$. Note that (M,F′) and (N,F″) are 1-dimensional $\mathcal{X}$-derived submanifolds of $(\mathbb{R}^2,F)$. Moreover, one can see that the 0-dimensional submanifold $\{(0,0)\}$ of $\mathbb{R}^2$ admits the $\mathcal{X}$-derived structure induced by F which is given by the zero bicontravariant tensor field. ■

Let now F be a left (right) involutive bicontravariant tensor field on M of constant rank k. From the Frobenius theorem it follows that the distribution $\mathcal{E}^a(F)$ ($\mathcal{E}^b(F)$) is integrable, and so, it defines a foliation of M of dimension k called the *left* (*right*) *F-foliation* of M. If F is involutive and balanced, then the left F-foliation of M coincides with the right one, and we call such a foliation the *F-foliation* of M. Give attention that if F is involutive, then the left and right F-foliations of M are different in general. The simplest example of such F-foliations is defined by the bicontravariant tensor field $F = (\partial/\partial x)\otimes(\partial/\partial y)$ on $\mathbb{R}^2$.

We say that an $\mathcal{X}$-derived manifold (M,F) is *left involutive, right involutive, involutive, balanced* and of *constant rank*, respectively, if so is the bicontravariant tensor field F.

Let (M,F) be a balanced involutive $\mathcal{X}$-derived manifold of constant rank. If L is a leaf of the F-foliation of M, we conclude from the Frobenius theorem that every vector field from $\mathfrak{M}^a(F)$ or $\mathfrak{M}^b(F)$ is tangent to L. Hence and from Corollary 4.6 we get

4.10. Corollary. *Let* (M,F) *be a balanced involutive $\mathcal{X}$-derived manifold of constant rank. If* L *is a leaf of the* F*-foliation of* M*, then* L *admits the $\mathcal{X}$-derived structure induced by* F. ■

Let (M,F) and (N,G) be $\mathcal{X}$-derived manifolds. Consider the Cartesian product $M \times N$ of differentiable manifolds. For any $q \in N$ ($p \in M$) let $i_q: M \longrightarrow M \times N$ ($j_p: N \longrightarrow M \times N$) be the smooth map defined by $i_q(x) = (x,q)$ ($j_p(y) = (p,y)$). Define the bicontravariant tensor field H on $M \times N$ by

$$H_{(p,q)} = i_{q*}F_p + j_{p*}G_q.$$

We call H the *flat product* of F and G and write $H = F \,\square\, G$. The $\mathcal{X}$-derived manifold $(M \times N, F \,\square\, G)$ is said to be the *flat product* of (M,F) and (N,G) which will be also denoted by $(M,F) \,\square\, (N,G)$. Let π_1 (π_2) be the projection from $M \times N$ onto the first (second) factor of $M \times N$. Note that π_1 and π_2 define $\mathcal{X}$-derived morphisms from $(M \times N, F \,\square\, G)$ onto (M, F) and (N,G), respectively. One can see that the biderivation $(\cdot,\cdot)_H$ of the algebra $C^\infty(M \times N)$ is a unique biderivation of this algebra satisfying the following conditions:

$$(\pi_1^*\alpha',\pi_1^*\beta')_H = \pi_1^*(\alpha',\beta')_F \text{ for all } \alpha',\beta' \in C^\infty(M);$$

$$(\pi_2^*\alpha'',\pi_2^*\beta'')_H = \pi_2^*(\alpha'',\beta'')_G \text{ for all } \alpha'',\beta'' \in C^\infty(N);$$

$$(\pi_1^*\alpha',\pi_2^*\alpha'')_H = (\pi_2^*\alpha'',\pi_1^*\alpha')_H = 0 \text{ for all } \alpha' \in C^\infty(M),\ \alpha'' \in C^\infty(N).$$

The following example shows that in general $(M \times N,\ F \,\square\, G)$ is not a product of (M,F) and (N,G) in the category of $\mathcal{X}$-derived manifolds.

4.11. Example. Let $M = N = \mathbb{R}$ and $F = G = (\partial/\partial x)\otimes(\partial/\partial x)$. Then $(M \times N,\ F \,\square\, G) = (\mathbb{R}^2,H)$ where $H = (\partial/\partial x)\otimes(\partial/\partial x) + (\partial/\partial y)\otimes(\partial/\partial y)$. On the other hand, $H' = H + (\partial/\partial x)\otimes(\partial/\partial y)$ is also an $\mathcal{X}$-derived structure on $\mathbb{R}^2$ such that the projections π_1 and π_2 define $\mathcal{X}$-derived morphisms from $(\mathbb{R}^2,H')$ onto $(\mathbb{R},F)$ and $(\mathbb{R},G)$, respectively. If $(\mathbb{R}^2,H)$ would be a product of $(\mathbb{R},F)$ and $(\mathbb{R},G)$ in the category of $\mathcal{X}$-derived manifolds, we conclude from the universal property of such a product that the identity map of $\mathbb{R}^2$ defines an $\mathcal{X}$-derived morphism from $(\mathbb{R}^2,H')$ to $(\mathbb{R}^2,H)$, which is impossible. ∎

We shall regard the flat product $\square$ as a bifunctor from the category of pairs of $\mathcal{X}$-derived manifolds to the category of $\mathcal{X}$-derived manifolds. One can see that this bifunctor makes the last category to be (relaxed) commutatively monoidal (see [9]) with respect to the identity object given by an arbitrary one-point $\mathcal{X}$-derived manifold $(E,0)$ and the following list of canonical $\mathcal{X}$-derived isomorphisms:

$$((M,F) \,\square\, (N,G)) \,\square\, (P,H) \cong (M,F) \,\square\, ((N,G) \,\square\, (P,H));$$
$$(E,0) \,\square\, (M,F) \cong (M,F) \cong (M,F) \,\square\, (E,0);$$
$$(M,F) \,\square\, (N,G) \cong (N,G) \,\square\, (M,F),$$

where (M,F), (N,G) and (P,H) are arbitrary $\mathcal{X}$-derived manifolds. We think to be clear the sense of these isomorphisms and remark only that they are defined by the canonical bijections for the corresponding Cartesian products of underlying sets. Note that if

$$f\colon (M,F) \longrightarrow (M',F') \text{ and } g\colon (N,G) \longrightarrow (N',G')$$

are morphisms of $\mathcal{X}$-derived manifolds, then the assignment $(x,y) \mapsto (f(x),g(y))$ for $(x,y) \in M \times N$ defines the $\mathcal{X}$-derived morphism

$$f \,\square\, g\colon (M,F) \,\square\, (N,G) \longrightarrow (M',F') \,\square\, (N',G').$$

If in addition $f'\colon (M',F') \longrightarrow (M'',F'')$ and $g'\colon (N',G') \longrightarrow (N'',G'')$ are morphisms of $\mathcal{X}$-derived manifolds too, we have

$$(f' \,\square\, g')\circ(f \,\square\, g) = (f'\circ f) \,\square\, (g'\circ g).$$

One can see that the category of symmetric (antisymmetric) $\mathcal{X}$-derived manifolds and the category of Poisson manifolds are monoidal subcategories of the category of $\mathcal{X}$-derived manifolds. A class $\mathcal{M}$ of $\mathcal{X}$-derived manifolds is said to be *monoidal* in case $(M,F) \in \mathcal{M}$ and $(N,G) \in \mathcal{M}$ involve $(M,F) \,\square\, (N,G) \in \mathcal{M}$. It is seen that $\mathcal{M}$ is monoidal if and only if it defines a full monoidal subcategory of the category of $\mathcal{X}$-derived manifolds.

An $\mathcal{X}$-derived manifold (M,F) is said to be *nondegenerate, quasi-nondegenerate, regular, left (right) strictly regular, strictly regular, left (right) strictly coregular* and *strictly coregular,* respectively, if so is F (Section 3). Note that if $\mathcal{M}$ is a corresponding class of such manifolds, then it is determined by properties of dimension functions. This means that an $\mathcal{X}$-derived manifold (M,F) belongs to $\mathcal{M}$ provided that the dimension functions ρ_F, δ^a_F, δ^b_F, τ^a_F, and τ^b_F have distinguished properties. For example, if $\mathcal{M}$ is the class of all left strictly regular $\mathcal{X}$-derived manifolds, then an $\mathcal{X}$-derived manifold (M,F) belongs to $\mathcal{M}$ provided that $\tau^b_F(x) = \dim M$ for each $x \in M$ (Theorem 3.15).

By an easy verification we get

4.12. Proposition. *If* (M,F) *and* (N,G) *are $\mathcal{X}$-derived manifolds, then for each* $(x,y) \in M \times N$ *the following conditions hold:*

(1) $\rho_{F\square G}(x,y) = \rho_F(x) + \rho_G(y)$;

(2) $\delta^a_{F\square G}(x,y) = \delta^a_F(x) + \delta^a_G(y)$ $(\delta^b_{F\square G}(x,y) = \delta^b_F(x) + \delta^b_G(y))$;

(3) $\tau^a_{F\square G}(x,y) = \tau^a_F(x) + \tau^a_G(y)$ $(\tau^b_{F\square G}(x,y) = \tau^b_F(x) + \tau^b_G(y))$. ∎

This proposition immediately implies

4.13. Corollary. *The classes consisting of all $\mathcal{X}$-derived manifolds which are nondegenerate, quasi-nondegenerate, regular, left (right) strictly regular, strictly regular, left (right) strictly coregular and strictly coregular, respectively, are monoidal.* ∎

As we know, there are monoidal classes of $\mathcal{X}$-derived manifolds which are not determined by properties of dimension functions, namely, such classes are represented by symmetric (antisymmetric) $\mathcal{X}$-derived manifolds and by Poisson manifolds, respectively. Moreover, one can see that there are other ones, for example, the classes consisting of all $\mathcal{X}$-derived manifolds which are left (right) involutive, involutive and balanced, respectively.

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