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LIFTINGS OF 1-FORMS TO THE p^r-VELOCITIES BUNDLE

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Our starting point are notions introduced by Morimoto [2],[3] and the classification of liftings to the higher order tangent bundle made by Gancarzewicz and Mahi [1]. We want to classify all linear liftings of 1-forms to p^r-velocities bundle. We deduce that every lifting is linear combination over R of Morimoto's liftings and o,i-liftings(introduced in this paper). Further we will assume that all considered objects are smooth (of class C^∞).

1. Preliminaries

In this section we present the definition of lifting of 1-forms and some related basic facts.

Let M be a smooth manifold. Denote by $T^{(r,p)}M$ the set of r-jets at $0 \in R^p$ of mappings from R^p to M. It forms bundle over M called p^r-velocities bundle. The mapping $\pi: T^{(r,p)}M \longrightarrow M$ is the bundle projection.

$$\pi(j_0^r\gamma) = \gamma(0).$$

Every chart (U, x^i) on M induces the chart $(\pi^{-1}(U), x^{i,\nu})$ on $T^{(r,p)}M$, where i is an integer number between 0 and dim(M), ν is an element of N^p such that $|\nu| \leq r$. The induced chart is given by

$$x^{i,\nu}(j_0^r\gamma) = \frac{1}{\nu!}D^\nu(x^i \circ \gamma)(0). \tag{1.1}$$

Now we present the definition of lifting of 1-forms to the p^r-velocities bundle.

Definiton 1.2. *A mapping*

$$\mathcal{L}: \mathcal{X}^*(M) \longrightarrow \mathcal{X}^*(T^{(r,p)}(M)),$$

[0]This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\mathcal{X}^*(M)$ and $\mathcal{X}^*(T^{(r,p)}(M))$ are the modules of 1-forms on M and on $T^{(r,p)}M$, is called lifting of 1-forms from M to $T^{(r,p)}M$ if following conditions hold:

(a) $\mathcal{L}$ is linear over R, that is, for every 1-forms ω, $\omega\prime$ on M and every real numbers a, b

$$\mathcal{L}(a\omega + b\omega\prime) = a\mathcal{L}(\omega) + b\mathcal{L}(\omega\prime)$$

(b) $\mathcal{L}$ is local, that is, for every open subset $U \subset M$ and for every 1-forms ω, $\omega\prime$ on M such that $\omega_{|U} = \omega\prime_{|U}$

$$\mathcal{L}(\omega)_{|\pi^{-1}(U)} = \mathcal{L}(\omega\prime)_{|\pi^{-1}(U)},$$

(c) $\mathcal{L}$ is natural, that is, for every diffeomorphism $\phi: U \longrightarrow V$ of open sets $U, V \subset M$ and for every 1-form ω

$$\mathcal{L}(\phi^*\omega) = (T^{(r,p)}\phi)^*\mathcal{L}(\omega),$$

where * denotes the pull-back of 1-form,

(d) $\mathcal{L}$ is regular, that is, for every open set $K \subset R^k$ and for every smooth mapping $\omega: K \times M \longrightarrow T^*M$, the induced mapping

$$K \times T^{(r,p)}M \ni (t,p) \longrightarrow (\mathcal{L}\omega_t)(p) \in T^*(T^{(r,p)}M)$$

is smooth.

The proposition below is the simple conclusion from Definition 1.2.

Proposition 1.3 *Let $\mathcal{L}$ be a lifting of 1-forms from M to $T^{(r,p)}M$. For any 1-form ω and for any vector field X on M*

$$\mathcal{L}(L_X\omega) = L_{X^c}(\mathcal{L}\omega).$$

Let define notion of (λ)-lifting(see: [2]). Let f be a function defined on M, $f \in C^\infty(M)$. (λ)-lifting of f(denoted by $f^{(\lambda)}$) is a function on $T^{(r,p)}M$ given as follows:

$$f^{(\lambda)}(j_0^r\gamma) = \frac{1}{\lambda!}D_\lambda(f \circ \gamma)(0). \tag{1.4}$$

Immediately from (1.1) and (1.4) it's clear that

$$x^{i,\nu} = (x^i)^{(\nu)}. \tag{1.5}$$

Lemma 1.6 *For any $\lambda \in N^p$: $|\lambda| \leq r$ there exists one and only one mapping $L_\lambda: \mathcal{X}^*(M) \longrightarrow \mathcal{X}^*(T^{(r,p)}M)$ satisfying the following condition*

$$L_\lambda(fdg) = \sum_{\nu \leq \lambda} f^{(\nu)} dg^{(\lambda-\nu)},$$

where $\nu \leq \lambda$ means, that for any $i = 1 \dots p$ $\nu_i \leq \lambda_i$. Proof of Lemma 1.6 is analogous to considerations in [2]. The mapping constructed in Lemma 1.6 is called (λ)-lifting of 1-forms. $L_\lambda(\omega)$ will be denoted by $\omega^{(\lambda)}$.

Theorem 1.7 *For every $\lambda \in N^p$ such that $|\lambda| \leq r$ the mapping*

$$(\lambda): \mathcal{X}^* \ni \omega \longrightarrow \omega^{(\lambda)} \in \mathcal{X}^*(T^{(r,p)}M)$$

is a lifting of 1-forms to $T^{(r,p)}M$ in meaning of Definition 1.2

Now we define just another type of liftings to $T^{(r,p)}M$. Let $\pi_{1,i}$ be a projection from $T^{(r,p)}M$ to TM defined as follows

$$\pi^r_{1,i}(j^r_0\gamma) = \dot{\bar{\gamma}}(0)), \tag{1.8}$$

where $\bar{\gamma}: (-\varepsilon, \varepsilon) \longrightarrow M$ is a curve derived from γ by formula

$$\bar{\gamma}(t) = \gamma(0, \dots, t, \dots, 0).$$

For any 1-form ω on M and for any integer number $i = 1, \dots, p$ we can define 1-form $\omega^{o,i}$ by

$$\omega^{o,i} = d(\omega \circ \pi^r_{1,i}). \tag{1.9}$$

Theorem 1.10 *For every $1, \dots, p$ the mapping*

$$()^{o,i}: \mathcal{X}^*(M) \ni \omega \longrightarrow \omega^{o,i} \in \mathcal{X}^*(T^{(r,p)}M)$$

is a lifting of 1-forms from M to $T^{(r,p)}M$.

Proof: Directly from (1.9) the mapping $()^{o,i}$ is linear, local and regular. For every open sets $U, V \subset M$ and for every diffeomorphism $\phi: U \longrightarrow V$ we have:

$$d\phi \circ \pi^r_{1,i} = \pi^r_{1,i} \circ T^{(r,p)}\phi.$$

Therefore by standard check the mapping $()^{o,i}$ is natural.

2. Classification of liftings to the p^r-velocities bundle

In this section we formulate the main result . It is classification of all liftings from M to the p^r-velocities bundle. We present several lemmas and propositions useful for proof of the main theorem.

Lemma 2.1(*see: [1]*) *Let $f\colon R^k \longrightarrow R$ be a differantiable function.*
(a). If f satisfies the condition

$$\sum_{j=1}^{k} v^j \frac{\partial f}{\partial v^j} = 0$$

then f is constant
(b). If f satisfies the condition

$$\sum_{j=1}^{k} v^j \frac{\partial f}{\partial v^j} + f = 0$$

then f is identically zero on R^k.

Lemma 2.2(*see: [1]*) *Let (U, x^i) be a chart on M and x_0 be a point of U. If ω is a closed 1-form on M, then there exists a vector field X on M such that*

$$\omega = L_X(dx^1) \tag{2.3}$$

in some neibhborhood of x_0.

Lemma 2.3 *Let (U, x^i) be a chart on M. We denote by $(\pi^{-1}(U), x^{i,\nu})$ the induced chart on $T^{(r,p)}M$. Then*
a).

$$L_{x^j \frac{\partial}{\partial x^i}} dx^k = \delta^i_k dx^j,$$

b).

$$(x^j \frac{\partial}{\partial x^i})^C = \sum_{|\mu|\le r} x^{j,\mu} \frac{\partial}{\partial x^{i,\mu}},$$

c). for every function f on $\pi^{-1}(U)$

$$L_{(x^j \frac{\partial}{\partial x^i})^C}(f dx^{k,\nu}) = \sum_{|\mu|\le r} x^{j,\mu} \frac{\partial f}{\partial x^{i,\mu}} dx^{k,\nu} + \delta^i_k f dx^{j,\nu}.$$

Proof:
ad a). The local vector field $x^j \frac{\partial}{\partial x^i}$ is generated by the one-parameter group of transformations ψ_t given by

$$\phi_t(x) = \phi^{-1}(x^1, \dots, tx^j + x^i, \dots, x^n),$$

where (ψ, U) is a chart on M, $\phi = (x^1, \dots, x^n)$.

$$L_{(x^j \frac{\partial}{\partial x^i})^C}(dx^k) = \lim_{t\to 0} \frac{1}{t}(dx^k - (\psi_t)_*(dx^k)) =$$

$$= \lim_{t\to 0} \frac{1}{t}(dx^k - dx^k \circ d\psi_{-t}) = \lim_{t\to 0} \frac{1}{t}(dx^k - d(-tx^j\delta^i_k + x^k)) =$$

$$= \lim_{t\to 0} \frac{1}{t}(t\delta^i_k dx^j) = \delta^i_k dx^j$$

ad b). The mapping $T^{(r,p)}\psi_t$ is the one-parameter group of transformations of $(x^j \frac{\partial}{\partial x^i})^C$. Let $j^r_0(\gamma)$ be an element of $T^{(r,p)}M$.

$$T^{(r,p)}\psi_t(j^r_0\gamma) = (j^r_0(\phi^{-1}(\gamma^1, \dots, t\gamma^j + \gamma^i, \dots, \gamma^n)),$$

where $\gamma^k = (\phi \circ \gamma)^k$. Let calculate value of $x^{k,\nu}$ on the above jet. From (1.1) we have

$$x^{k,\nu}(j^r_0(\phi^{-1}(\gamma^1, \dots, t\gamma^j + \gamma^i, \dots, \gamma^n)) = \frac{1}{\nu!}D^\nu(\gamma^k + t\delta^i_k\gamma^j) =$$

$$= \frac{1}{\nu!}D^\nu(\gamma^k) + t\delta^i_k\frac{1}{\nu!}D^\nu(\gamma^j) = x^{k,\nu}(j^r_0\gamma) + t\delta^i_k x^{j,\nu}(j^r_0\gamma)$$

The (k,ν)-coordinate of $T^{(r,p)}\psi_t$ is equal $x^{k,\nu} + t\delta^i_k x^{j,\nu}$ and if $i \neq k$ this coordinate doesn't depend on t, therefore

$$(x^j \frac{\partial}{\partial x^i})^C = \sum_{|\mu|\le r} x^{j,\mu}\frac{\partial}{\partial x^{i,\mu}},$$

ad c). Let f be a function on $\pi^{-1}(U)$.

$$L_{(x^j \frac{\partial}{\partial x^i})^C}(f dx^{k,\nu}) =$$

$$L_{(x^j \frac{\partial}{\partial x^i})^C}(f) \cdot dx^{k,\nu} + f \cdot L_{(x^j \frac{\partial}{\partial x^i})^C} dx^{k,\nu}$$

From Proposition 1.3

$$L_{(x^j \frac{\partial}{\partial x^i})^C} dx^{k,\nu} = (L_{x^j \frac{\partial}{\partial x^i}} dx^k)^{(\nu)}.$$

Using a). and (1.5) we obtain

$$L_{(x^j \frac{\partial}{\partial x^i})^C} dx^{k,\nu} = \delta^i_k dx^{j,\nu}$$

Now we calculate $L_{(x^j \frac{\partial}{\partial x^i})^C}(f)$.

$$L_{(x^j \frac{\partial}{\partial x^i})^C}(f) = df((x^j \frac{\partial}{\partial x^i})^C) =$$

$$= df(\sum_{|\mu|\le r} x^{j,\mu}\frac{\partial}{\partial x^{i,\mu}}) = \sum_{|\mu|\le r} x^{j,\mu}\frac{\partial f}{\partial x^{i,\mu}}.$$

Now the proof is finished.

The proposition below provides classification of liftings for closed 1-forms on M.

Proposition 2.4 *Let M be a manifold. If $\mathcal{L}$ is a lifting of 1-forms to the p^r-velocities bundle, then there exist real numbers c_ν, where $\nu \in N^p : |\nu| \le r$ such that for every closed 1-form ω on M*

$$\mathcal{L}(\omega) = \sum_{|\nu|\le r} c_\nu \omega^{(\nu)}.$$

Proof: Let (U, x^i) be a chart on M. Then 1-form $\mathcal{L}(dx^1)$ on $T^{(r,p)}M$ in local coordinates is given by

$$\mathcal{L}(dx^1) = \sum_{k=1}^{n}\sum_{|\nu|\le r} a_{k,\nu}dx^{k,\nu}, \tag{2.5}$$

where $a_{k,\nu}$ are functions on $\pi^{-1}(U)$. From Lemma 2.3 a)

$$L_{x^j\frac{\partial}{\partial x^i}} dx^k = \delta^i_k dx^j. \tag{2.6}$$

Using Proposition 1.3 we obtain

$$\delta^i_k \mathcal{L}(dx^j) = L_{(x^j\frac{\partial}{\partial x^i})^C}\mathcal{L}(dx^k). \tag{2.7}$$

For $k = 1$ from (2.7) we have

$$\delta^1_i \mathcal{L}(dx^j) = L_{(x^j\frac{\partial}{\partial x^i})^C}\mathcal{L}(dx^1)$$

Next from (2.5) the following formula is valid

$$\delta^1_i \mathcal{L}(dx^j) = \sum_{k=1}^{n}\sum_{|\nu|\le r} L_{(x^j\frac{\partial}{\partial x^i})^C}(a_{k,\nu}dx^{k,\nu}).$$

Applying Lemma 2.3 c) to $f = a_{k,\nu}$ we obtain

$$\delta^1_i \mathcal{L}(dx^j) = \sum_{k=1}^{n}\sum_{|\nu|\le r}(\sum_{|\mu|\le r} x^{j,\mu}\frac{\partial a_{k,\nu}}{\partial x^{i,\mu}}dx^{k,\nu} + \delta^i_k a_{k,\nu}dx^{j,\nu}) =$$

$$= \sum_{k=1}^{n} \sum_{|\nu|\le r} \Big(\sum_{|\mu|\le r} x^{j,\mu} \frac{\partial a_{k,\nu}}{\partial x^{i,\mu}} + \delta^i_k a_{k,\nu} \Big) dx^{k,\nu}. \tag{2.8}$$

From (2.8) and (2.5) we have

$$\delta^1_i a_{k,\nu} = \sum_{|\mu|\le r} x^j_\mu \frac{\partial a_{k,\nu}}{\partial x^{i,\mu}} + \delta^j_k a_{i,\nu}. \tag{2.9}$$

For $i = j = k = 1$ it gives

$$\sum_{|\mu|\le r} x^{1,\mu} \frac{\partial a_{1,\nu}}{\partial x^{1,\mu}} = 0. \tag{2.10}$$

Applying (2.8) to $i = j \neq 1,\ k = 1$ we obtain

$$\sum_{|\mu|\le r} x^{j,\mu} \frac{\partial a_{1,\nu}}{\partial x^{j,\mu}} = 0. \tag{2.11}$$

Formulas (2.10) and (2.11) together give the following condition

$$\sum_{j=1}^{n} \sum_{|\mu|\le r} x^{j,\mu} \frac{\partial a_{1,\nu}}{\partial x^{j,\mu}} = 0.$$

According to Lemma 2.1 $a_{1,\nu}$ is constant for every $\nu \in N^p$. From (2.9) for $i \neq 1,\ k = j = 1$ we obtain

$$a_{i,\nu} = - \sum_{|\mu|\le r} x^{1,\mu} \frac{\partial a_{1,\nu}}{\partial x^{i,\mu}} = 0.$$

Let denote by c_ν the constant value of $a_{1,\nu}$. Then from previous considerations we can write $\mathcal{L}(dx^k)$ in the form

$$\mathcal{L}(dx^k) = \sum_{|\nu|\le r} c_\nu dx^{1,\nu}.$$

From Lemma 2.2 for every closed 1-form ω there exists a vector field X such that $\omega = L_X(dx^1)$. Therefore

$$\mathcal{L}(\omega) = \mathcal{L}(L_X dx^1) = L_{X^c}(\mathcal{L}(dx^1)) =$$

$$= L_{X^c}\Big(\sum_{|\nu|\le r} c_\nu dx^{1,\nu}\Big) = \sum_{|\nu|\le r} c_\nu L_{X^c}(dx^1)^{(\nu)} = \sum_{|\nu|\le r} c_\nu (L_X dx^1)^{(\nu)} =$$

$$= \sum_{|\nu|\le r} c_\nu \omega^{(\nu)}.$$

Now the proof is finished.

The main result can be expressed in the following theorem.

Theorem 2.5 *Let M be a manifold such that* $dim(M) \geq 2$. *If* $\mathcal{L}$ *is a lifting of 1-forms from M to the* p^r*-velocities bundle then* $\mathcal{L}$ *is a linear combination over R of* (λ)*-liftings and* o, i*-liftings, that is, there exist real numbers* c_ν, $\nu \in N^p\colon |\nu| \leq r$ *and* $c_{o,i}$, $i = 1, \ldots, p$ *such that for every 1-form* ω *on M we have*

$$\mathcal{L}(\omega) = \sum_{|\nu|\leq r} c_\nu \omega^{(\nu)} + \sum_{i=1}^{p} c_{o,i}\omega^{o,i}$$

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