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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 30. pp. [95]--100.

Persistent URL: http://dml.cz/dmlcz/701509

NATURAL AFFINORS ON THE EXTENDED r-TH ORDER TANGENT BUNDLES

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The extended r-th order tangent bundle $E^r M$ over an n-dimensional manifold M is defined as dual vector bundle $E^r M = (J^r(M, \mathbf{R}))^*$. The r-th order tangent bundle $T^{(r)}M = (J^r(M, \mathbf{R})_0)^*$ over M is a vector subbundle of $E^r M$ and we have a natural decomposition $E^r M = T^{(r)}M \times \mathbf{R}$. For $r = 1$ we obtain the time-dependent tangent bundle $E^1 M = TM \times \mathbf{R}$.

In this paper we determined all natural affinors (i.e. tensor fields of type $(1,1)$) on E^r. In item 3 we defined geometrically four natural affinors on E^r. Then we prove that all natural affinors on E^r are their linear combinations, the coefficient of which are arbitrary smooth functions on $\mathbf{R}$. For $r = 1$ we rededuce a special case of another general result by M. Doupovec and the second authors, [2].

All manifolds and maps are assumed to be infinitely differentiable.

1. Let M be a manifold. The vector bundle $E^r M = (J^r(M, \mathbf{R}))^*$ is called *extended r-th order tangent bundle*. The target map $\beta : J^r(M, \mathbf{R}) \to \mathbf{R}$ can be interpreted as a vector bundle epimorphism of $J^r(M, \mathbf{R})$ onto the 1-dimensional vector bundle $M \times \mathbf{R}$ which admits a splitting defined by the r-jets of the constant functions on M. Hence $ker\beta = J^r(M, \mathbf{R})_0$ is a vector subbundle of $J^r(M, \mathbf{R})$ such that $J^r(M, \mathbf{R}) = ker\beta \times \mathbf{R}$. The vector bundle $T^{(r)}M = (ker\beta)^*$ is called *r-th order tangent bundle* over M. This is a vector subbundle of $E^r M$ and we have a natural decomposition $E^r M = T^{(r)}M \times \mathbf{R}$, provided we have used the canonical identification of $\mathbf{R}$ with $\mathbf{R}^*$.

Every smooth map $f : M \to N$ induces a linear map

$$J^r_{f(x)}(N, \mathbf{R}) \ni j^r_{f(x)}\varphi \to j^r_x(\varphi \circ f) \in J^r_x(M, \mathbf{R})$$

[0]This paper is in final form and no version of it will be submitted for publication elsewhere.

$x \in M$, $\varphi : N \to \mathbf{R}$. The transposed linear maps $E^r_x M \to E^r_{f(x)} N$ determine a vector bundle homomorphism $E^r f : E^r M \to E^r N$ covering f. One verifies easily that the rule $M \to E^r M$, $f \to E^r f$ is a bundle functor on the category of all manifolds in the sense of [5]. Since $E^r f(T^{(r)}M) \subset T^{(r)}N$ for every $f : M \to N$ and pullbacks of constant functions are constant functions, we have $E^r f = T^{(r)} f \times id_{\mathbf{R}}$ under the decomposition $E^r M = T^{(r)}M \times \mathbf{R}$.

2. An affinor on a manifold M is a tensor field of type $(1,1)$ on M which can be interpreted as a vector bundle homomorphism $TM \to TM$ covering the identity on M. Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds, see e. g. [4], [5]. According to [6], *a natural affinor* on $\mathcal{F}$ is a system of affinors $Q_M : T(\mathcal{F}M) \to T(\mathcal{F}M)$ on $\mathcal{F}(M)$, for every n-manifold M, satisfying the condition

$$T(\mathcal{F}f) \circ Q_M = Q_N \circ T(\mathcal{F}f)$$

for every local diffeomorphism $f : M \to N$.

Our problem is to find all natural affinors on the restriction of E^r to the category of n-manifolds and their local diffeomorphisms.

3. First we define four natural affinors on E^r.

I. Let $\delta_M : T(T^{(r)}M) \to T(T^{(r)}M)$ be the identity map. By means of the decomposition $T(E^r M) = T(T^{(r)}M) \times T\mathbf{R}$, $\delta = \{\delta_M\}$ induces a natural affinor $\widetilde{\delta} = \{\widetilde{\delta}_M\}$ on E^r.

II. Analogously, the identity affinor $\delta_{\mathbf{R}} : T\mathbf{R} \to T\mathbf{R}$ on $\mathbf{R}$ induces a natural affinor $\widetilde{\delta}_{\mathbf{R}}$ on E^r. Let us observe that $\widetilde{\delta} + \widetilde{\delta}_{\mathbf{R}}$ is the identity affinor on E^r.

III. Let $y \in T^{(r)}M$ and $x = \pi(y) \in M$. There is the natural isomorphism $\psi_y : V_y(T^{(r)}M) \to (T^{(r)}M)_x$ between the vertical space $V_y(T^{(r)}M) = T_y(T^{(r)}M)_x)$ and the fiber $(T^{(r)}M)_x$ of $T^{(r)}M$ over x. The jet projection $\beta_1 : J^r(M,\mathbf{R})_0 \to J^1(M,\mathbf{R})_0$ induce an inclusion $i_M : TM = T^1 M \to T^{(r)}M$. Now we define a linear map $V_{M,y} : T_y(T^{(r)}M) \to T_y(T^{(r)}M)$ as the composition

$$T_y(T^{(r)}M) \xrightarrow{T_y\pi} T_{\pi(y)}M \xrightarrow{i_M} (T^{(r)}M)_{\pi(y)} \xrightarrow{\psi_y^{-1}} V_y(T^{(r)}M) \subset T_y(T^{(r)}M)$$

Let $V_M : T(T^{(r)}M) \to T(T^{(r)}M)$ be defined by $V_M | T_y(T^{(r)}M) = V_{M,y}$ for any $y \in T^{(r)}M$. The system $V = \{V_M\}$ is a natural affinor on $T^{(r)}$ which induces a natural affinor $\widetilde{V}$ on E^r.

IV. Let L_M be the Liouville vector field on $T^{(r)}M$, i.e. the vector field determines by the homotheties. This is a natural vector field on $T^{(r)}M$. Then the system $L \otimes dt = \{L_M \otimes dt\}$ is a natural affinor on E^r, where t is the canonical coordinate on $\mathbf{R}$.

Theorem. *All natural affinors on E^r are linear combinations of $\widetilde{\delta}$, $\widetilde{\delta}_{\mathbf{R}}$, $\widetilde{V}$ and $L \otimes dt$, the coefficients of which are arbitrary smooth functions on* $\mathbf{R}$.

The proof will occupy the rest of the paper.

4. By the general theory, [5], it is sufficient to study the linear maps of the standard fiber $T(E^r\mathbf{R})$ over $0 \in \mathbf{R}^n$ into itself. We write $x = (x^i) \in \mathbf{R}^n$, $t \in \mathbf{R}$, $y = (y_1, \dots, y_r) \in T^{(r)}\mathbf{R}^n$, where $y_s = (y^{i_1 \dots i_s})$, are the induced coordinates on $T^{(r)}\mathbf{R}^n$, [7]. The additional coordinates in $T(E^r\mathbf{R}^n)$ are given by $X^i = dx^i$, $T = dt$, $Y^{i_1 \dots i_s} = dy^{i_1 \dots i_s}$. Then any linear map of the standard fiber $T(E^r\mathbf{R})$ over $0 \in \mathbf{R}^n$ into itself has the following form

$$
\begin{aligned}
\overline{X}^i &= a^i_j(t,y)X^j + b^i(t,y)T + \sum_{s=1}^{r} c^i_{i_1 \dots i_s}(t,y)Y^{i_1 \dots i_s} \\
\overline{T} &= A_j(t,y)X^j + B(t,y)T + \sum_{s=1}^{r} C_{i_1 \dots i_s}(t,y)Y^{i_1 \dots i_s} \qquad (1)\\
\overline{Y}^{i_1 \dots i_s} &= \alpha^{i_1 \dots i_s}_j(t,y)X^j + \beta^{i_1 \dots i_s}(t,y)T + \sum_{p=1}^{r} \gamma^{i_1 \dots i_s}_{j_1 \dots j_p}(t,y)Y^{j_1 \dots j_p}
\end{aligned}
$$

where the coefficients are arbitrary smooth function in t and y. Let us remark that the equivariant maps corresponding to the natural affinors $\widetilde{\delta}$, $\widetilde{\delta}_{\mathbf{R}}$, $\widetilde{V}$, $L \otimes dt$ are:

$$
\begin{array}{rlll}
\widetilde{\delta}: & \overline{X}^i = X^i, & \overline{T} = 0, & \overline{Y}^{i_1 \dots i_s} = Y^{i_1 \dots i_s} \\
\widetilde{V}: & \overline{Y}^i = X^i, & \overline{T} = 0, & \overline{Y}^{i_1 \dots i_s} = 0 \\
\widetilde{\delta}_{\mathbf{R}}: & \overline{X}^i = 0, & \overline{T} = Y, & \overline{Y}^{i_1 \dots i_s} = 0 \\
L \otimes dt: & \overline{X}^i = 0, & \overline{T} = Y, & \overline{Y}^{i_1 \dots i_s} = y^{i_1 \dots i_s} T
\end{array}
$$

5. First, we consider the equivariancy of (1) with respect to the homotheties $\bar{x}^i = kx^i$, $k \neq 0$. We have $\bar{t} = t$, $\bar{y}^{i_1 \dots i_s} = y^{i_1 \dots i_s}$, $\overline{X}^i = kX^i$, $\overline{T} = T$ and $\overline{Y}^{i_1 \dots i_s} = k^s Y^{i_1 \dots i_s}$.

The equivariancy of the first row of (1) implies

$$
\begin{aligned}
a^i_j(t, ky_1, k^2y_2, \dots, k^r y_r) &= k a^i_j(t, y_1, y_2, \dots, y_r) \\
b^i(t, ky_1, k^2y_2, \dots, k^r y_r) &= k b^i(t, y_1, y_2, \dots, y_r) \\
k^{s-1} c^i_{i_1 \dots i_s}(t, ky_1, k^2y_2, \dots, k^r y_r) &= c^i_{i_1 \dots i_s}(t, y_1, y_2, \dots, y_r)
\end{aligned}
$$

By the homogenous function theorem we obtain

$$
\begin{aligned}
a^i_j(t, y_1, y_2, \dots, y_r) &= a^i_j(t, y_1) \\
b^i(t, y_1, y_2, \dots, y_r) &= b^i(t, y_1) \qquad (2)\\
c^i_{i_1}(t, y_1, y_2, \dots, y_r) &= c^i_{i_1}(t)
\end{aligned}
$$

are functions of the indicated variable only. Moreover, it holds

$$c^i_{i_1\dots i_s}(t, y_1, y_2, \dots, y_r) = 0 \qquad \text{for } s > 1 \tag{3}$$

The equivariancy of the second row of (1) implies

$$\begin{aligned} kA_j(t, ky_1, k^2y_2, \dots, k^ry_r) &= A_j(t, y_1, y_2, \dots, y_r) \\ B(t, ky_1, k^2y_2, \dots, k^ry_r) &= B(t, y_1, y_2, \dots, y_r) \\ k^sC_{i_1\dots i_s}(t, ky_1, k^2y_2, \dots, k^ry_r) &= C_{i_1\dots i_s}(t, y_1, y_2, \dots, y_r) \end{aligned}$$

Letting $k \to 0$ we obtain

$$A_j = 0, \qquad B(t, y_1, \dots, y_r) = B(t), \qquad C_{i_1\dots i_s} = 0 \tag{4}$$

In the end, the equivariancy of the last row of (1) implies

$$\begin{aligned} \alpha^j_i(t, ky_1, k^2y_2, \dots, k^ry_r) &= k^{s-1}\alpha^j_i(t, y_1, y_2, \dots, y_r) \\ \beta^{i_1\dots i_s}(t, ky_1, k^2y_2, \dots, k^ry_r) &= k^s\beta^{i_1\dots i_s}(t, y_1, y_2, \dots, y_r) \\ \gamma^{i_1\dots i_s}_{j_1\dots j_p}(t, ky_1, k^2y_2, \dots, k^ry_r) &= k^{s-p}\gamma^{i_1\dots i_s}_{j_1\dots j_p}(t, y_1, y_2, \dots, y_r) \end{aligned}$$

Hence

$\alpha^{i_1\dots i_s}_i$ is a function of $t, y_1, \dots, y_{s-1}$

$\beta^{i_1\dots i_s}$ is a function of $t, y_1, \dots, y_s$

$\gamma^{i_1\dots i_s}_{j_1\dots j_p}$ is a function of $t, y_1, \dots, y_{s-p}$ if $p \le s$

$\gamma^{i_1\dots i_s}_{j_1\dots j_p} = 0 \qquad$ if $p > s$

Since the coefficients in $\overline{T}$ are independent on y, for every $t \in \mathbf{R}$ the functions $b^i(t, y)$, $\beta^{i_1\dots i_s}(t, y)$ defines an equivariant map of $(T^{(r)}\mathbf{R}^n)_0$ into itself. According to a result of the second author and G. Vosmanská, [7], such natural transformations are homotheties. This implies

$$b^i(t, y) = b(t)y^i, \qquad \beta^{i_1\dots i_s}(t, y) = b(t)y^{i_1\dots i_s} \tag{6}$$

where b is a smooth function on $\mathbf{R}$.

From (2) — (6) we now deduce that (1) can be written in the form

$$\begin{aligned} \overline{X}^i &= a^i_j(t, y_1)X^j + b(t)y^iT + c^i_j(t)Y^j \\ \overline{T} &= B(t)T \\ \overline{Y}^{i_1\dots i_s} &= \alpha^{i_1\dots i_s}_j(t, y_1, \dots, y_{s-1})X^j + b(t)y^{i_1\dots i_s}T \\ &\quad + \sum_{p=1}^{s} \gamma^{i_1\dots i_s}_{j_1\dots j_p}(t, y_1, \dots, y_{s-p})Y^{j_1\dots j_p} \end{aligned} \tag{7}$$

These relations read that for any $p < r$ the subspace $(TE^p\mathbf{R}^n)_0$ is invariant with respect to our equivariant map. It means that the natural affinor Q under consideration induces a natural affinor $\overline{Q}$ on E^p, $p < r$.

6. To finish the proof we will use the induction with respect to r.

If $r = 1$, our theorem represents a special case of a result by M. Doupovec and the second author, [2], for $E^1M = TM \times \mathbf{R}$.

Assume that the theorem is true for $r-1$. Let Q be a natural affinor on E^r. By the remark from the end of item 5, Q defines a natural affinor on E^{r-1}. The induction hypothesis and (7) imply that the corresponding equivariant map can be written in the form

$$\begin{aligned} \overline{X}^i &= a(t)X^i + b(t)y^iT + c(t)Y^i \\ \overline{T} &= B(t)T \\ \overline{Y}^{i_1\dots i_s} &= b(t)y^{i_1\dots i_s}T + a(t)Y^{i_1\dots i_s} \qquad \text{if } s < r \\ \overline{Y}^{i_1\dots i_r} &= \alpha_j^{i_1\dots i_r}(t, y_1, \dots, y_{r-1})X^j + b(t)y^{i_1\dots i_r}T \\ &\quad + \sum_{p=1}^{r} \gamma_{j_1\dots j_p}^{i_1\dots i_r}(t, y_1, \dots, y_{r-p})Y^{j_1\dots j_p} \end{aligned} \tag{8}$$

From the equivariancy of (7) with respect to the transformations

$$\overline{x}^i = x^i + K^i_{i_1\dots i_r}x^{i_1}\dots x^{i_r}$$

$K^i_{i_1\dots i_r} \in \mathbf{R}$, we deduce by a standard evaluation

$$\begin{aligned} \alpha_i^{i_1\dots i_r} &= 0 \\ \gamma_{j_1\dots j_p}^{i_1\dots i_r} &= 0 \qquad \text{if } p < r \\ \gamma_{j_1\dots j_r}^{i_1\dots i_r}Y^{j_1\dots j_r} &= a(t)Y^{i_1\dots i_r} \end{aligned}$$

Thus we have

$$\begin{aligned} \overline{X}^i &= a(t)X^i + b(t)y^iT + c(t)Y^i \\ \overline{T} &= B(t)T \\ \overline{Y}^{i_1\dots i_s} &= b(t)y^{i_1\dots i_s}T + a(t)Y^{i_1\dots i_s} \qquad \text{for } s = 1, \dots, r \end{aligned}$$

This means that the affinor Q has the following form $a(t)\widetilde{\delta} + B(t)\widetilde{\delta}_{\mathbf{R}} + c(t)\widetilde{V} + b(t)L\otimes dt$. This completes the proof.

7. From our theorem we can deduce immediately the complete characterization of natural affinors on $T^{(r)}$. Namely, we have

Corollary. *All natural transformations on $T^{(r)}$ are $k_1\delta + k_2V$, where $k_1, k_2 \in \mathbf{R}$, δ is the identity affinor and V is the natural affinor defined in item 3.*

This result can be deduced immediately from results by M. Doupovec [1].

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