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1^0. *GELFAND-ZETLIN SUBALGEBRA*

Let $\mathcal{G}=\mathcal{GL}(n,\mathbb{C})$, e_{ij} be the matrix units. Consider the standard inclusions $\mathcal{G}_1\subset\mathcal{G}_2\subset\ldots\subset\mathcal{G}_n=\mathcal{G}$ where $\mathcal{G}_k=\langle e_{ij}\,|\,i,j=1..k\rangle$. Denote U_k the universal enveloping algebra of $\mathcal{G}_k$, Z_k the center of U_k, $U=U_n$ and put $\Gamma=\langle Z_k\,|\,k=1..n\rangle$, the *Gelfand-Zetlin (GZ-) subalgebra* of U. One knows [6] that Z_k is the polynomial ring in k variables c_{kj} $(j=1..k)$ where $c_{kj}=\Sigma e_{i_1i_2}e_{i_2i_3}\ldots e_{i_ji_1}$ for all sequences $(i_1i_2\ldots i_j)$ with $i_s\in\{1..k\}$; moreover, Γ is the polynomial ring in $n(n+1)/2$ variables c_{kj} $(k=1..n,\ j=1..k)$.

Proposition 1. Γ is a maximal commutative subalgebra in U.

Proof. Following the known proof of the Harish-Chandra theorem ([2],2.5.7), one can show that $u\in\Gamma$ if and only if $\rho(u)\in\rho(\Gamma)$ for any finite-dimensional representation ρ of $\mathcal{G}$. But the Gelfand-Zetlin formulae [6] imply that $\rho(\Gamma)$ coincides with the set of all diagonal matrices. Hence if u commutes with all $a\in\Gamma$ then $\rho(u)\in\rho(\Gamma)$ which accomplishes the proof.

A $\mathcal{G}$-module V will be called a *GZ-module* provided $V=\oplus_\chi V(\chi)$ where χ runs through the space $\Gamma^\wedge$ of characters of Γ and $V(\chi)=\{v\in V\,|\ \forall a\in\Gamma\ \exists m\ (a-\chi(a))^m v=0\}$. Denote $\mathfrak{G}$ the category of all GZ-modules (it contains all finite-dimensional $\mathcal{G}$-modules [6]). For $V\in\mathfrak{G}$ put $\mathrm{supp}V=\{\chi\in\Gamma^\wedge\ |\ V(\chi)\neq 0\}$, $V_\chi=\{v\in V\ |\ \forall a\in\Gamma\ av=\chi(a)v\}$.

Consider another polynomial ring L in $n(n+1)/2$ variables

l_{kt} $(k=1..n,\ t=1..k)$ and the homomorphism $\iota:\Gamma\to L$ which maps c_{kj} to $\Sigma_t l_{kt}{}^j \Pi_{p\neq t}(1-(l_{kt}-l_{kp})^{-1})$. The symmetric group S_k acts on L permuting l_{kt} $(t=1..k)$. Thus the direct product $S=\Pi_k S_k$ acts on L.

Proposition 2. ι is an inclusion and its image coinsides with the ring of invariants L^S.

Proof. It is easy to check that $\iota(c_{kj})$ is a symmetric polynomial in l_{kt} $(t=1..k)$ of the form $\Sigma_t l_{kt}{}^j+f$ with $\deg f<j$. As the power sums are algebraically independent and generate L^S, it proves the statement.

From now on we identify Γ with its image in L. It is convenient to choose new generators $\sigma_{kj}=\sigma_j(l_{k1},\dots,l_{kk})$ where σ_j are the elementary symmetric functions. The inclusion ι induces the surjection $\pi:L^\wedge\to\Gamma^\wedge$ which identifies $\Gamma^\wedge$ with $S\backslash L^\wedge$. For any $\lambda\in L^\wedge$ we shall write $V(\lambda)$ instead of $V(\pi(\lambda))$ and put $\lambda_{kt}=\lambda(l_{kt})$, $\sigma_{kj}(\lambda)= =\sigma_j(\lambda_{k1},\dots,\lambda_{kk})=\lambda(\sigma_{kj})$.

Denote $X_k{}^+=e_{k,k+1}$, $X_k{}^-=e_{k+1,k}$. Then the set $\{X_k{}^\pm \mid k=1..n-1\}$ generates U. Of course, $X_k{}^\pm$ commutes with elements of Z_t if $t\neq k$. Define the polynomials $f_{kjm}{}^\pm(\sigma_{k1},\dots,\sigma_{kj})$ by the formula:

$$F_{kj}{}^\pm(T)=\Pi_t(T-\sigma_{kj}(\lambda\pm\delta_{kt}))=T^k+\Sigma_m f_{kjm}{}^\pm(\sigma_{k1}(\lambda),\dots,\sigma_{kk}(\lambda))T^m$$

where δ_{kt} $(t=1..k)$ are given by the rule: $\delta_{kt}(l_{qp})=1$ if $q=k,p=t$ and 0 otherwise.

Proposition 3. (i) $X_k{}^\pm V(\lambda)\subset\Sigma_t V(\lambda\pm\delta_{kt})$ for any $\lambda\in L^\wedge$.

(ii) $\sigma_{kj}{}^k X_k{}^\pm+\Sigma_m\sigma_{kj}{}^m X_k{}^\pm f_{kjm}{}^\pm(\sigma_{k1},\dots,\sigma_{kj})=0$.

(iii) If V is a simple GZ-module, $\lambda\in L^\wedge$ and $\lambda_{kt}-\lambda_{kp}\notin\mathbb{Z}$ for all indeces $t\neq p$, then $V(\lambda)=V_\lambda$.

Proof. If V is finite-dimensional, *(i)* is known [6]. Moreover, in this case $V(\lambda)=V_\lambda$. Thus $F_{kj}{}^\pm(\sigma_{kj})X_k{}^\pm v=0$ for any $v\in V$. This means that the left part of the equality *(ii)* annihilates V. By the Harish-Chandra theorem it proves *(ii)*. *(i)* and *(iii)* are easy consequences of *(ii)*.

Corollary 1. If $V(\chi)\neq 0$ for some $\chi\in\Gamma^\wedge$ and the module V is simple, then V is a GZ-module.

Corollary 2. If V is an indecomposable GZ-module and $V(\lambda)\neq 0$ for some $\lambda\in L^\wedge$ then $\operatorname{supp}V\subset\pi(\lambda+\Delta)$ where Δ is the subgroup of $L^\wedge$ generated by all δ_{kj} $(k=1..n-1,\ j=1..k)$.

Obviousely, if P and P' are cosets modulo Δ, then either $\pi(P)=\pi(P')$ or $\pi(P)\cap\pi(P')=\emptyset$. For $D=\pi(P)$ denote $\mathfrak{G}_D$ the complete subcategory of $\mathfrak{G}$ consisting of all modules V with $suppV\subset D$ and $\equiv$ the equivalence relation on $\Gamma^\wedge$: $\chi\equiv\chi'$ provided both of them belong to $\pi(P)$ for some coset $P\in L^\wedge/\Delta$.

Corollary 3. $\mathfrak{G}=\bigsqcup_{D\in\Gamma^\wedge/\equiv}\mathfrak{G}_D$.

For $\lambda\in L^\wedge$ put $\sigma_{kj,ip}=\sigma_{kj}(\lambda+\delta_{ki}-\delta_{ip})$ and define the polynomials $g_{kjm}(\sigma_{k1},\dots,\sigma_{kj})$ by the formula:

$$G_{kj}(T)=\prod_{i\neq p}(T-\sigma_{kj,ip})=T^K+\Sigma_m g_{kjm}(\sigma_{k1}(\lambda),\dots,\sigma_{kj}(\lambda))T^m$$

where $K=k(k-1)/2$.

Proposition 4. For any $a\in\Gamma$ the element

$$Y_{kj}(a)=\sigma_{kj}{}^K X_k{}^- aX_k{}^+ +\Sigma_m\sigma_{kj}{}^m X_k{}^- aX_k{}^+ g_{kjm}(\sigma_{k1},\dots,\sigma_{kj})$$

belongs to Γ and the same is true if we permute $^+$ and $^-$.

The *proof* is quite analogous to that of *proposition 3*.

2^0. *GELFAND-ZETLIN CATEGORIES*

For $\chi\in\Gamma^\wedge$ denote $I_\chi=Ker\chi$, Γ_χ the I_χ-adique completion of Γ. Consider for any pair $\chi,\psi\in\Gamma^\wedge$ the Γ-bimodule $U(\psi,\chi)=\{u\in U \mid \forall m\ \exists n\ I_\psi{}^n u\subset UI_\chi{}^m\}$. The adique topologies of Γ induce a topology on $U(\psi,\chi)$, so we can form its completion $\mathfrak{F}(\psi,\chi)$. Now we can define a category $\mathfrak{F}_D$ for $D\in\Gamma^\wedge/\equiv$ whose objects are characters $\chi\in D$ and sets of morphisms are $\mathfrak{F}(\psi,\chi)$. Surely, the category $\mathfrak{G}_D$ is equivalent to the category $\mathfrak{F}_D$-*mod* of $\mathfrak{F}_D$-modules, i.e. continuous linear functors from $\mathfrak{F}_D$ to the category of vector spaces over $\mathbb{C}$ with discrete topologies. The following result is a simple corollary of the *abstract nonsence*.

Proposition 5. Denote $\mathfrak{F}_\chi=\mathfrak{F}(\chi,\chi)$ and $\mathcal{F}_\chi=\mathfrak{F}_\chi/rad\mathfrak{F}_\chi$. Then:

(i) If V is a simple GZ-module with $V(\chi)\neq 0$, then $V(\chi)$ is a simple $\mathcal{F}_\chi$-module.

(ii) For any simple $\mathcal{F}_\chi$-module M there exists the unique (up to isomorphism) simple GZ-module V with $V(\chi)\simeq M$.

Denote $\nu(D)$ (resp. $\nu(\chi)$) the number of non-isomorphic simple GZ-modules V with $suppV\subset D$ (resp. $V(\chi)\neq 0$). Define two open dense subsets Ω_1,Ω_2 of $\Gamma^\wedge$ in the following way:

$$\Omega_1=\{\chi=\pi(\lambda) \mid \lambda_{ki}-\lambda_{kp}\notin\mathbb{Z} \text{ for all } i\neq p \text{ and } k=2..n-1\}$$

$$\Omega_2=\Omega_1\cap\{\chi=\pi(\lambda) \mid \lambda_{ki}-\lambda_{k-1,p}\notin\mathbb{Z} \text{ for all } i,p \text{ and } k=2..n\}$$

Remark that both Ω_1 and Ω_2 are stable under the relation $\equiv$.

THEOREM 1. (i) If $\chi\in\Omega_1$, then $\nu(\chi)=1$ and $\dim V(\chi)=1$ for the unique simple GZ-module V with $V(\chi)\neq 0$.

(ii) If $D\subset\Omega_2$, then $\nu(D)=1$.

Proof. Let $\chi=\pi(\lambda)$ and $\chi\in D$. Denote $x_{ki}^{\pm}$ the morphism from $\mathfrak{F}(\chi,\pi(\lambda+\delta_{ki}))$ generated by $X_k^{\pm}$. Of course, if χ runs through D, the images of Γ_χ and all possible $x_{ki}^{\pm}$ generate $\mathfrak{F}_D$. If $\chi\in\Omega_1$, *proposition 3(iii)* implies that the image of Γ_χ in $\mathcal{F}_\chi$ consists only of scalars: the class of $a\in\Gamma$ coinsides with that of $\chi(a)$. Let $y_{ki}(a)$ for any $a\in\Gamma$ be the image in $\mathcal{F}_\chi$ of $\Sigma_i x_{ki}^{-} a x_{ki}^{+}$. We know then from *proposition 4* that $G_{kj}(\sigma_{kj}(\lambda))y_{kj}(a)$ is also a scalar, namely, the image of $Y_{kj}(a)$. If $\chi\in\Omega_1$, then $G_{kj}(\sigma_{kj}(\lambda))\neq 0$, hence $y_{kj}(a)$ is also a scalar. Putting $a=1,\sigma_{k2},\dots,\sigma_{kk}$ we obtain k linear equations for k products $x_{ki}^{-}x_{ki}^{+}$ and one can check that their determinant is not 0 provided $\lambda_{ki}\neq\lambda_{kp}$ for $i\neq p$. Thus these products are scalars too. The same is true for $x_{ki}^{+}x_{ki}^{-}$. But it is easy to see that all these products generate $\mathcal{F}_\chi$ (together with Γ_χ). Hence $\mathcal{F}_\chi$ is either 0 or $\mathbb{C}$ and $\nu(\chi)$ is respectively either 0 or 1. But using GZ-formulae as in [6] one can construct a GZ-module W with $W(\chi)\neq 0$ for all $\chi\in D$ which proves *(i)*. The same formulae show that if $D\subset\Omega_2$, then W is simple and hence it is the only simple GZ-module in $\mathfrak{G}_D$ which proves *(ii)*.

Remark. It follows from *proposition 3* that if V is a simple GZ-module and $v\in V(\chi)$ then in any case $\sigma_{k1}v=\chi(\sigma_{k1})v$, $\sigma_{nj}v=\chi(\sigma_{nj})v$ and $(\sigma_{kj}-\chi(\sigma_{kj}))^k v=0$ in other cases.

Conjectures. (i) $0<\nu(\chi)<\infty$ for any $\chi\in\Gamma^\wedge$ and $\nu(D)<\infty$ for any $D\in\Gamma^\wedge$.

(ii) $\dim V(\chi)\leq n$ for any simple GZ-module and any $\chi\in\Gamma^\wedge$.

(iii) The image of Γ in $\mathrm{End}_{\mathbb{C}}V(\chi)$ is a maximal commutative subalgebra and coinsides with the subalgebra generated by a Jordan cell (where V and χ are as in (ii)).

These conjectures are true if $n\leq 3$. Really, if $n=2$, it follows from [3] that $\nu(\chi)=1$ and $\dim V(\chi)=1$ for all V and χ; $\nu(D)$ can equals $1,2$ or 3; if $\nu(D)=3$, then one of the simple modules in $\mathfrak{G}_D$ is finite-dimensional and all finite-dimensional simple $\mathcal{G}$-modules are of this type (cf. also [2], 7.0.9).

If $n=3$, the following statements hold (cf. [4]).

THEOREM 2. For any $\chi\in\Gamma^\wedge$ *and any* $D\in\Gamma^\wedge/\equiv$

(i) $0<\nu(\chi)\leq 2$ *and* $\nu(D)<\infty$.

(ii) If V is a simple GZ-module with $V(\chi)\neq 0$, *then* $\dim V(\chi)\leq 2$ *and if* $\dim V(\chi)=2$, *then* $\nu(\chi)=1$.

(iii) If $\dim V(\chi)=2$, *then* σ_{22} *acts on* $V(\chi)$ *as a Jordan cell.*

Proof. A straitforward calculation using the results of [1] shows that $\mathcal{F}_\chi$ equals either $\mathbb{C}$ or $\mathbb{C}\oplus\mathbb{C}$ or $M_2(\mathbb{C})$. According to *proposition 5* it proves *(ii)*. If all objects of the category $\mathfrak{G}_D$ are isomorphic, then it implies also *(i)*. Otherwise, if V is a simple module from $\mathfrak{G}_D$, then $V(\chi)=0$ for some $\chi\in D$. But it follows from [5] that there exist only finitely many such modules in $\mathfrak{G}_D$ which accomplishes the proof of *(i)*. The statement *(iii)* can be easily checked.

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