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THE RELATION BETWEEN THE DUAL AND THE ADJOINT RADON TRANSFORMS

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ABSTRACT The Radon transform and its dual can be linked directly by two operators from geometrical nature, taking into account the difference between the measures used on Euclidean space and on the set of hyperplanes. Using the relation between the dual and the adjoint of the Radon transform if considered as operator between L_2-spaces, several results regarding continuity, compactness and singular values decompositions can be generalized.

1 Introduction and notation

Let $\mathbf{R}^m$ be the m-dimensional Euclidean space with the inner product $\langle\ ,\ \rangle$ and let $\mathbf{P}^m$ be the set of hyperplanes in $\mathbf{R}^m$. Each hyperplane σ can be represented by a couple $(\vec{\theta}, p)$ where $\vec{\theta}$ is a unit vector in $\mathbf{R}^m$ orthogonal to σ and $p\vec{\theta} \in \sigma$. Notice that $(-\vec{\theta}, -p)$ represents the same hyperplane. Hence we can consider a function on $\mathbf{P}^m$ as an even function on $S^{m-1} \times \mathbf{R}$, where S^{m-1} is the unit sphere of $\mathbf{R}^m$.
Let f be a measurable function on $\mathbf{R}^m$. The Radon transform of f is defined as

$$\mathcal{R}f(\vec{\theta}, p) = \int_\sigma f(\vec{x}) d\vec{x}$$

as far as the integral exists, where σ is the hyperplane represented by $(\vec{\theta}, p)$. For a measurable function g on $\mathbf{P}^m$ the dual Radon transform is given by

$$\mathcal{R}^\# g(\vec{x}) = \int_{S^{m-1}} g(\vec{\theta}, \langle\vec{x}, \vec{\theta}\rangle) dS_{\vec{\theta}},$$

again as far as the integral exists.
For a domain Ω in $\mathbf{R}^m$ we use the classical notation $L_2(\Omega, w)$ for the weighted L_2-space with inner product

$$(f, g) = \int_\Omega \overline{f}.g.w d\vec{x}$$

[0]This paper is in its final form and no version of it will be submitted for publication elsewhere.

where w is a non-negative function on Ω. For a subset I of $\mathbf{R}$ with weight function v the notation $L_2(K, v)$ is used where $K = S_{m-1} \times I$ and the inner product is given by

$$(f,g) = \int_{S^{m-1}} dS_{\vec{\theta}} \int_I \overline{f(\vec{\theta},p)} g(\vec{\theta},p) v(p) dp.$$

It is well known that $\mathcal{R}$ as operator from $L_2(\mathbf{R}^m, 1)$ to $L_2(S^{m-1} \times \mathbf{R}, 1)$ is a closable, densely defined operator. If we denote by $\mathcal{R}^*$ the operator given by

- $g \in \mathrm{dom}(\mathcal{R}^*) \iff \mathcal{R}^\# g(\vec{x})$ exists for $\vec{x}$ a.e. in $\mathbf{R}^m$.
- $(\mathcal{R}^* g)(\vec{x}) = \mathcal{R}^\# g(\vec{x})$

then the closure of $\mathcal{R}^*$ is the adjoint of $\mathcal{R}$ (see [3] and [4]).

2 The relation between the Radon transform and its dual

In projective geometry a dual mapping between points and hyperplanes is given by mapping a point $\vec{x}$ with homogeneous coordinates $(x_0, x_1, \ldots, x_m)$ to the hyperplane $\tau(\vec{x})$ where the coordinates of $\vec{x}$ become the coefficients in the equation of the hyperplane

$$\tau(\vec{x}) : x_1 y_1 + \ldots + x_m y_m = x_0 y_0$$

(one could also choose a minus sign at the right hand side, but the plus sign is more convenient for our purposes). If we restrict τ to the the affine space we get a bijection between $\mathbf{R}_0^m$ and the set of hyperplanes not going through the origin which shall be denoted by $\mathbf{P}_0^m$. As a dual mapping it has the property

$$\vec{x} \in \tau(\vec{y}) \iff \vec{y} \in \tau(\vec{x}).$$

With the coordinates $(\vec{\theta}, p)$ used for elements of $\mathbf{P}_0^m$ we get

$$\begin{aligned} \tau(\vec{x}) &= \left(\frac{\vec{x}}{|\vec{x}|}, \frac{1}{|\vec{x}|}\right) \\ \tau^{-1}(\vec{\theta}, p) &= \frac{\vec{\theta}}{p}. \end{aligned}$$

Since $\mathcal{R}^\#$ consists of an integration over the hyperplanes through $\vec{x}$ it is clear that we can relate $\mathcal{R}^\#$ to $\mathcal{R}$ using τ if we introduce a scalar correction to take into account the difference between the measures used on $\mathbf{R}_0^m$ and $\mathbf{P}_0^m$. This will be done introducing the operators K_1 and K_m as follows:

- Let f be an arbitrary function on $\mathbf{R}_0^m$. Then $K_1 f$ is defined on $\mathbf{P}_0^m$ by
$$K_1 f(\tau(\vec{x})) = f(\vec{x}).|\vec{x}|.$$
- Let g be an arbitrary function on $\mathbf{P}_0^m$. Then $K_m g$ is defined on $\mathbf{R}_0^m$ by
$$K_m g(\tau^{-1}(\vec{\theta}, p)) = g(\vec{\theta}, p).|p|^m.$$

It is clear that $K_m K_1 f(\vec{x}) = f(\vec{x}).|\vec{x}|^{1-m}$

Theorem 2.1

$$K_1\mathcal{R}^{\#} = 2\mathcal{R}K_m.$$

Proof.
Let f be a function on $\mathbf{P}_0^m$ and take a hyperplane $(\vec{\theta}, p)$ such that $\mathcal{R}K_m f(\vec{\theta}, p)$ exists. Then

$$\begin{aligned}\mathcal{R}K_m f(\vec{\theta}, p) &= \int_{\vec{\theta}^\perp} K_m f(\vec{y} + p\vec{\theta})d\vec{y} \\ &= \int_{\vec{\theta}^\perp} f\left(\frac{\vec{y}+p\vec{\theta}}{|\vec{y}+p\vec{\theta}|}, \frac{1}{|\vec{y}+p\vec{\theta}|}\right)\frac{1}{|\vec{y}+p\vec{\theta}|^m}d\vec{y}\end{aligned}$$

introducing the variable $\vec{\eta} = \frac{\vec{y}+p\vec{\theta}}{|\vec{y}+p\vec{\theta}|}$ gives $\langle\vec{\eta}, \vec{\theta}\rangle = \frac{p}{|\vec{y}+p\vec{\theta}|}$ and $d\vec{y} = \frac{p^{m-1}}{\langle\vec{\eta},\vec{\theta}\rangle^m}dS$ and so

$$\begin{aligned}\mathcal{R}K_m f(\vec{\theta}, p) &= \frac{1}{2}\int_{S^{m-1}} f\left(\vec{\eta}, \left\langle\eta, \frac{\vec{\theta}}{p}\right\rangle\right)\frac{\langle\vec{\eta},\vec{\theta}\rangle^m}{p^m}\frac{p^{m-1}}{\langle\vec{\eta},\vec{\theta}\rangle^m}dS \\ &= \frac{1}{2p}\mathcal{R}^{\#} f\left(\frac{\vec{\theta}}{p}\right) \\ &= \frac{1}{2}K_1\mathcal{R}^{\#} f(\vec{\theta}, p).\end{aligned}$$

∎

3 The relation between the dual and the adjoint Radon transforms

As mentioned before $\mathcal{R}^{\#}$, restricted as an operator between $L_2(S^{m-1} \times \mathbf{R}, 1)$ to $L_2(\mathbf{R}^m, 1)$ is, up to operator closure, the adjoint of $\mathcal{R}$ itself, i.e.

$$[\mathcal{R}f, g]_1 = (f, \mathcal{R}^{\#}g)_1$$

if $\mathcal{R}^{\#}g$ is in $L_2(\mathbf{R}^m, 1)$. This result can be extended to the case where $\mathcal{R}$ is considered as an operator from $L_2(\Omega, w)$ to $L_2(\mathcal{R}(\Omega), v)$ for more general domains Ω and weight functions w and v. In this case we denote the adjoint of $\mathcal{R}$ as $\mathcal{R}_w^{\#}$ and we have for suitable f and g that

$$\begin{aligned}[\mathcal{R}f, g]_v &= [\mathcal{R}f, vg]_1 \\ &= (f, \mathcal{R}^*vg)_1 \\ &= \left(f, \frac{1}{w}\mathcal{R}^{\#}vg\right)_w. \qquad (1)\end{aligned}$$

If we introduce the notation W for the operator mapping a function f on Ω to wf and V for the analogical operator on the image space we get

$$\mathcal{R}_w^{\#} = \overline{W^{-1}\mathcal{R}^{\#}V} \tag{2}$$

at least if w and v are chosen so that the classes of suitable functions in (1) are not too restricted. We shall not give general criteria for this but prove (1) in each case taken separately. Finally we get the following scheme:

$$\begin{array}{ccccc}
L_2(\Omega, w) & \xrightarrow{\quad W \quad} & L_2\left(\Omega, \frac{1}{w}\right) & \xrightarrow{\quad \frac{K_1}{2} \quad} & L_2\left(\tau(\Omega), \frac{p^{1-m}}{w(p^{-1})}\right) \\
\uparrow \mathcal{R}_w^{*} & & \uparrow \mathcal{R}^{\#} & & \uparrow \mathcal{R} \\
L_2(\mathcal{R}(\Omega), v) & \xrightarrow{\quad V \quad} & L_2\left(\mathcal{R}(\Omega), \frac{1}{v}\right) & \xrightarrow{\quad K_m \quad} & L_2\left(\tau^{-1}(\mathcal{R}(\Omega)), \frac{r^{m-1}}{v(r^{-1})}\right)
\end{array}$$

where W, V, K_m and $\frac{K_1}{2}$ are isometries. Indeed, that W and V are isometries is elementary while e.g. for K_m we have that

$$\begin{aligned}
(f,g)_{L_2(\mathcal{R}(\Omega),\frac{1}{v})} &= \int_{S^{m-1}} dS \int_{-\infty}^{+\infty} \frac{1}{v(p)} \overline{f(\vec{\theta},p)} g(\vec{\theta},p) dp \\
&= \int_{S^{m-1}} dS \int_{-\infty}^{+\infty} \frac{1}{v(p^{-1})} \overline{K_m f(\vec{x}) r^m} K_m g(\vec{x}) r^m \frac{dr}{r^2} \\
&= (K_m f, K_m g)_{L_2\left(\tau^{-1}(\mathcal{R}(\Omega)), \frac{1}{v(p^{-1})} r^{m-1}\right)}.
\end{aligned}$$

the reasoning for K_1 follows the same line of thought.

4 Continuity estimates

In [1] we obtained the following continuity estimates for the Radon transform between Hilbert spaces (where we assume that Ω and the weight function w are invariant under $\mathrm{SO}(n)$, v depends only from p and $\vec{\theta}$ is an arbitrary unit vector): If the Radon transform of $\frac{1}{w}$ exists a.e. and if

$$K = \operatorname*{ess\,sup}_{p\in\mathbf{R}} \mathcal{R}\frac{1}{w}(\vec{\theta},p)v(p)$$

is finite, then the operator

$$\mathcal{R} : L_2(\Omega, w) \to L_2(\mathcal{R}(\Omega), v)$$

is continuous and has operator norm $\|\mathcal{R}\| \leq K$.
This implies the existence, if $\frac{1}{w}$ has a Radon transform, of an optimal weight function in the image space given by

$$v(p) = \left[\mathcal{R}\frac{1}{w}(\vec{\theta}, p)\right]^{-1}.$$

Since of course the continuity of $\mathcal{R}$ implies the continuity of $\mathcal{R}^*$ with the same operator norm we can derive new continuity estimates from the scheme.
case 1: Ω is equal to $\mathbf{R}^m$. Then $\tau^{-1}(\mathcal{R}(\Omega))$ and $\tau(\Omega)$ are equal to $\mathbf{R}_0^m$ and P_0^m respectively. If we put

$$u(\vec{x}) = \frac{|\vec{x}|^{m-1}}{v(|\vec{x}|^{-1})}$$

$$s(p) = \frac{p^{1-m}}{w(p^{-1})}$$

we get that $K_m^{-1}\frac{1}{w}$(which is independent of $\vec{\theta}$) is given by

$$K_m^{-1}\frac{1}{w}(p) = \frac{s(p)}{p}$$

and so, using theorem 2.1

$$\mathcal{R}\frac{1}{w}\left(\vec{\theta}, |\vec{x}|^{-1}\right) = K_1 \mathcal{R}^{\#}\left(\frac{s(p)}{p}\right)(\vec{x})$$

from which

$$K = \operatorname{ess\,sup}_t \mathcal{R}^{\#}\left(\frac{s(p)}{p}\right)(\vec{\theta}, t)\frac{|t|^{1-m}}{u(|t|^{-1})}$$

which gives us a criterion for the continuity of the Radon transform from $L_2(\mathbf{R}^m)$ to $L_2(\mathbf{P}^m)$ in terms of the weight functions u and s. This way some cases can be treated where K is not finite. Indeed, if v is continuous and non zero in 0 then the weight function u behaves like $|\vec{x}|^{m-1}$ at infinity, so $\frac{1}{u}$ does not have a Radon transform.
case 2: Ω different from $\mathbf{R}^m$. This case can be treated in a similar way by putting $w = +\infty$ outside Ω (and hence $s = 0$ outside $\tau(\Omega)$) It should be remarked however that $\tau^{-1}(\mathcal{R}(\Omega))$ reduces in any case to $\mathbf{R}_0^m$ itself or to the exterior of a ball, so the most interesting case will be where Ω is a ball of finite radius, which can be normalized to 1 of course) and hence $\tau^{-1}(\mathcal{R}(\Omega))$ is the exterior of the unit ball, which we shall denote by $\mathbf{E}^m$ for short in the sequel. For $\tau(\Omega)$, the set of hyperplanes not passing through the unit ball we shall use the short notation $\mathbf{T}^m$.

Examples

In [1] we proved continuity of the Radon transform in the cases

$$\mathcal{R} \;:\; L_2(\mathbf{R}^n, e^{r^2/2}) \to L_2(\mathbf{P}^m, e^{p^2/2})$$

$$\mathcal{R} \;:\; L_2\left(B(1), \left(1-r^2\right)^{-\alpha}\right) \to L_2\left(S^{m-1}\times[-1,1], \left(1-p^2\right)^{-\left(\alpha+\frac{m-1}{2}\right)}\right),$$

where $\alpha > -1$. From the smoothness of the weight functions considered it follows that our scheme is valid. Hence we get continuity of the Radon transform for the dual cases

$$\begin{aligned}\mathcal{R} &: L_2(\mathbf{R}^n, r^{m-1}e^{-1/2r^2}) \to L_2(\mathbf{P}^m, p^{1-m}e^{-1/2p^2}) \\ \mathcal{R} &: L_2\left(\mathbf{E}^m, (r^2-1)^{\alpha+\frac{m-1}{2}} r^{-2\alpha}\right) \to L_2\left(\mathbf{T}^m, \left(p^2-1\right)^{\alpha} p^{1-m-2\alpha}\right).\end{aligned}$$

Moreover for these cases we had a singular value decomposition, which was proved in [2] from which also compactness could be derived, so we have that also in the dual case a singular value decomposition exists and that the operator considered is compact. Explicitly we get for the case of the Radon transform between $L_2\left(\mathbf{E}^m, (r^2-1)^{\alpha+\frac{m-1}{2}}\right)$ and $L_2\left(\mathbf{T}^m, (p^2-1)^{\alpha} p^{1-m-2\alpha}\right)$ the singular value decomposition as follows: we start from the singular value decomposition of $\mathcal{R}$ between $L_2\left(B(1), (1-r^2)^{-\alpha}\right)$ and $L_2\left(S^{m-1} \times [-1,1], (1-p^2)^{-\left(\alpha+\frac{m-1}{2}\right)}\right)$ which is given by

$$\mathcal{R}\left(T_{nk}^{\alpha i}\right) = \frac{\pi^{\frac{m-1}{2}}\Gamma(\alpha+n+1)}{2^k\Gamma\left(\alpha+n+1+\frac{m-1}{2}+k\right)} S_{n+k,k}^{\alpha i}$$

where n is even and k arbitrary and the functions $T_{nk}^{\alpha i}$ and $S_{nk}^{\alpha i}$ are given by

$$T_{nk}^{\alpha i} = (1-|\vec{x}|^2)^{-\alpha} C_{nk}^{\alpha}(\vec{x}) H_k^{(i)}(\vec{x})$$

and

$$S_{nk}^{\alpha i}(\vec{\theta}, p) = \frac{(-1)^n 2^n \Gamma(\lambda+\frac{1}{2}+n)\Gamma(2\lambda)n!}{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)}(1-p^2)^{\lambda-\frac{1}{2}} C_n^{\lambda}(p)(-1)^{\frac{n+k}{2}} H_k^{(i)}(\vec{\theta})$$

($n+k$ even and $\lambda = \alpha + \frac{m}{2}$). and the singular values themselves are given by

$$(K_{nk}^{\alpha})^2 = \frac{2^{2\alpha+m-1}\pi^{m-1}(n+k)!\Gamma\left(\alpha+\frac{n}{2}+\frac{m}{2}+k\right)\Gamma\left(\alpha+\frac{n}{2}+1\right)}{\left(\frac{n}{2}\right)!\Gamma(2\alpha+m+n+k)\Gamma\left(\frac{m}{2}+k+\frac{n}{2}\right)}.$$

Hence the singular value decomposition of the adjoint transform $\mathcal{R}_w^*$ is given by

$$\begin{aligned}\mathcal{R}_w^*(S_{nk}^{\alpha i}) &= 0, & n < k \\ \mathcal{R}_w^*(S_{nk}^{\alpha i}) &= (K_{nk}^{\alpha})^2 \frac{2^k\Gamma\left(\alpha+n+\frac{m-1}{2}\right)}{\pi^{\frac{m-1}{2}}\Gamma(\alpha+n-k+1)} T_{n-k,k}^{\alpha i}, & n \geq k\end{aligned}$$

from the commuting scheme we derive that $2\mathcal{R}K_m V = K_1 W \mathcal{R}_W^*$ and hence for the functions

$$\begin{aligned}s_{nk}^{\alpha i} &= K_m V S_{nk}^{\alpha i}(\vec{x}) \\ &= \frac{(-1)^n 2^n \Gamma(\lambda+\frac{1}{2}+n)\Gamma(2\lambda)n!}{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)} C_n^{\lambda}\left(\frac{1}{|\vec{x}|}\right)(-1)^{\frac{n+k}{2}} H_k^{(i)}(\vec{\xi})|\vec{x}|^{-m}\end{aligned}$$

and

$$\begin{aligned} t_{nk}^{\alpha i} &= K_1 W T_{nk}^{\alpha i} \\ &= C_{nk}^{\alpha}(p) H_k^{(i)}(\vec{\theta})|p|^{-1} \end{aligned}$$

we get

$$\begin{aligned} \mathcal{R}(s_{nk}^{\alpha i}) &= 0\,, & n < k \\ \mathcal{R}(s_{nk}^{\alpha i}) &= (K_{nk}^{\alpha})^2 \frac{2^{k-1}\Gamma\left(\alpha+n+\frac{m-1}{2}\right)}{\pi^{\frac{m-1}{2}}\Gamma(\alpha+n-k+1)} t_{n-k,k}^{\alpha i}, & n \geq k \end{aligned}$$

(n even, k arbitrary). This generalizes the singular value decomposition given by Perry in [5] for the special case of $m = 2$. In a similar fashion we have for the second case, the Radon transform between $L_2(\mathbf{R}^n, r^{m-1}e^{-1/2r^2})$ and $L_2(\mathbf{P}^m, p^{1-m}e^{-1/2p^2})$ the following scheme:
the singular value decomposition of $\mathcal{R}$ between $L_2(\mathbf{R}^n, e^{r^2/2})$ and $L_2(\mathbf{P}^m, e^{p^2/2})$ is given by

$$\mathcal{R}\left(T_{nk}^{i}\right) = (-1)^k(\sqrt{2\pi})^{m-1} S_{n+k,k}^{i}.$$

where n is even and k arbitrary and the functions T_{nk}^i and S_{nk}^i are given by

$$T_{nk}^{i}(\vec{x}) = e^{\frac{-r^2}{2}} H_{n,m,k}(\vec{x}) H_k^{(i)}(\vec{x})$$

where again n is even and k arbitrary and

$$S_{nk}^{i}(\vec{\theta}, p) = e^{-\frac{p^2}{2}} H_n(p) H_k^{(i)}(\vec{\theta})$$

where $n + k$ is even. and the singular values themselves are given by

$$K_{nk}^2 = \frac{2^{\frac{m}{2}-n-k-\frac{1}{2}}\pi^{m-\frac{1}{2}}(n+k)!}{(\frac{n}{2})!\Gamma(\frac{m}{2}+k+\frac{n}{2})}.$$

The singular value decomposition of the adjoint transform $\mathcal{R}_w^*$ is given by

$$\begin{aligned} \mathcal{R}_w^*(S_{nk}^{i}) &= 0\,, & n < k \\ \mathcal{R}_w^*(S_{nk}^{i}) &= (K_{nk})^2 \frac{\left(\frac{n-k}{2}\right)!\Gamma\left(\frac{m}{2}+\frac{k}{2}+\frac{n}{2}\right)}{2^{\frac{m}{2}-n-2k-\frac{1}{2}}\pi^{\frac{3m-1}{2}} n!\Gamma\left(\frac{m}{2}\right)} T_{n-k,k}^{i}, & n \geq k. \end{aligned}$$

Hence for the functions

$$\begin{aligned} s_{nk}^{i} &= K_m V S_{nk}^{i}(\vec{x}) \\ &= H_n\left(\frac{1}{|\vec{x}|}\right) H_k^{(i)}(\vec{\xi})|\vec{x}|^{-m} \end{aligned}$$

and

$$t_{nk}^{i} = K_1 W T_{nk}^{i}$$

$$H_{n,m,k}\left(\frac{1}{p}\right) H_k^{(i)}(\vec{\theta})|p|^{-1}$$

we get

$$\begin{aligned} \mathcal{R}(s^i_{nk}) = &\ 0\ , & n < k \\ \mathcal{R}(s^i_{nk}) = &\ (K_{nk})^2 \frac{\left(\frac{n-k}{2}\right)!\Gamma\left(\frac{m}{2}+\frac{k}{2}+\frac{n}{2}\right)}{2^{\frac{m}{2}-n-2k+\frac{1}{2}}\pi^{\frac{3m-1}{2}} n!\Gamma\left(\frac{m}{2}\right)} t^i_{n-k,k}, & n \geq k \end{aligned}$$

(n even, k arbitrary).

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