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A *path structure* is a pair (M, S) where M is a differentiable manifold of class C^∞, S is a spray on M, $S : \dot{T}M \to T\dot{T}M$ is a second order differentiable equation on M with homogeneity condition of order 2:

$$[S, C] = S, \tag{1}$$

where $C : \dot{T}M \to T\dot{T}M$ denotes the Liouville vertical vector field on $\dot{T}M$. *

Without the assumption of the homogeneity (1) we speak of a *semi-spray*, or a *semi-path structure*,resp. If H is of class C^∞ on all TM, then the path structure is called as a *quadratic* one. A curve $\varphi : I \to M$ is a *path curve* of the spray when $\ddot{\varphi} = S \circ \dot{\varphi}$,i.e. φ is integral curve of the vector field S.

Consider now a connection structure in τ_M [2], or what is the same, a parallelism structure in τ_M. In a natural way, the autoparallel curves of the connection determine a path structure, called the geodesic spray of the connection. First we shall give an explicit expression of a geodesic path structure, and prove that the homogeneity of the connection implies the homogeneity of the geodesic path structure, and, specially, it will be clear that the spray of a linear connection is quadratic.

The inverse question is much more interesting: How to construct a connection, a parallelism compatible to a given path structure. *Compatibility* means here that the autoparallel curves of the connection just coincide the path curves of the spray. This question was answered by Ambrose, Palais, Singer [1] in the quadratic case. For general sprays J.Grifone [4] has given a construction, but his method does not produce a compatible connection for semi-sprays. Similar result is presented in [3] by M. Crampin. So it is open problem whether there exists a compatible connection for semi-sprays.

Searching the answer we slightly change the question as follows: Let be given a path structure (M, S) and a connection structure (M, H) over the same manifold which are com-

* This paper is in final form and no version of it will be submitted for publication elsewhere.

patible. Assuming the homogeneity of one of them, does the homogeneity of the other follow? We shall see later on that the homogeneity of a spray does not imply the homogeneity of the connection. We shall also discuss to what extent one of the compatible pair determines the other.

Geodesic spray of a connection structure

A *connection* H on a differentiable manifold M is just a splitting of the next short exact sequence:

$$0 \to V\dot{\tau}_M \to \tau_{\dot{T}M} \to \dot{\pi}_M(\tau_M) \to 0$$

where the notations denote the vertical bundle, the tangent bundle of $\dot{T}M$, the pull-back bundle of τ_M by $\dot{\pi}_M$, resp. Thus ImH is a Whitney summand of $V\dot{\tau}_M : \tau_{\dot{T}M} = V\dot{\tau}_M \oplus ImH$. Then a vector field $X \in \mathcal{X}(M)$ is parallel along a curve $\varphi : I \to M$ in M iff $dX \circ \dot{\varphi} = H(X \circ \varphi, \dot{\varphi})$, and φ is an autoparallel curve iff $\ddot{\varphi} = H(\dot{\varphi}, \dot{\varphi})$. ($\dot{\varphi}$ denotes the tangent curve of φ.) This latter relationship motivates the next

Lemma. For any connection map $H : \dot{\pi}_M(\tau_M) \to \tau_{\dot{T}M}$ the map defined by $S : \dot{T}M \to T\dot{T}M$

$$S(v) := H(v, v) \tag{2}$$

for all $v \in \dot{T}M$ is a semi-spray of M. The path curves of S are just the autoparallel curves of H.

Proof. Let $v \in \dot{T}M$ be an arbitrary tangent vector. Since

$$d\dot{\pi}_M(S(v)) = d\dot{\pi}_M(H(v, v)) = v,$$

S is a differentiable equation on M. φ is an autoparallel curve of H iff $\ddot{\varphi} = H(\dot{\varphi}, \dot{\varphi}) = S(\dot{\varphi})$, i.e. φ is a path curve of S.

The semi-spray S constructed here is called the *geodesic* semi-spray of the connection H.

A connection is called *homogeneous* when

$$H(\mu_t(v), w) = d\mu_t(H(v, w)) \quad (t \in R) \tag{3}$$

holds for any $v, w \in TM$ with $\pi_M(v) = \pi_M(w)$, where $\mu_t : TM \to TM$ is the multiplication in the fibers by $t \in R$. If H is supposed to be of class C^1 on all $TM \times TM$, then we speak of a linear connection.

Proposition. If a connection is homogeneous, then its geodesic semi-spray is a spray, and, consequently, the geodesic spray of a linear connection is a quadratic spray.

Proof. First we recall that (1) is equivalent to the assumption $S \circ \mu_t = t(d\mu_t \circ S)$ for all $t \in R$. Thus we are to verify this under the assumption of (3). For any $v \in TM, t \in R$

$$S(\mu_t(v)) = H(\mu_t(v), \mu_t(v)) = H(tv, tv) = tH(tv, v) = td\mu_t(H(v, v) = td\mu_t(S(v))$$

In the linear case the local parameters $G^i : \pi_M^{-1}(U) \to R$ of S are expressed as

$$G^i(x, y) = \Gamma^i_{jk}(x) y^j y^k$$

which is obviously of C^2.

Compatible connection and path structures

Definition. A path structure (M, S) and a connection structure (M, H) are called *strongly (weakly) compatible* if the autoparallel curves of H and the path curves of S coincide including their canonical parameters (apart from their parametrization).

It can be readily seen from the characterization of the distinguished curves that (M, S) and (M, H) are strongly compatible iff

$$S(v) = H(v, v) \tag{4}$$

for all $v \in TM$.

We recall that two sprays on M are called projectively equivalent if their path curves coincide apart from their parametrization. It is well know [5] that that S and $\bar{S}$ are projectively equivalent iff $S = \bar{S} + \lambda C$ where C is the Liouville vector field on TM. Therefore, it will be sufficient to investigate strongly compatibility.

Consider a compatible pair of a connection structure (M, H) and a path structure (M, S). We have seen that a connection structure determines uniquely a path structure, namely, its geodesic spray, furthermore, the homogeneity property of the connection is hereditary to its geodesic spray. Taking into account the fact that compatibility means that the values of S are just the values of H on the diagonal , S does not determine uniquely H.

In [1] there has been proved that given a quadratic spray S there is a unique linear torsion free connection with geodesic spray S. In this proof the expontial map of the spray is applied. J. Grifone [4] generalized this result for general sprays, using the Nijenhuis bracket formalism.

M. Crampin [3] also gave a procedure to construct a connection from a given semi-spray. His method does not supply a compatible connection in general, more exactly, the connection constructed from a semi-spray S in [3] is compatible to S iff S is a spray.

Finally, the following simple example shows us that the homogeneity of a spray does not imply the homogeneity of a compatible connection. In fact, considering a local coordinate system (x^i, y^i) over $\pi_M^{-1}(U) \in TM$, the compatibility relation (4) can be locally expressed as

$$G^i(x, y) = y^j \Gamma^i_j(x, y)$$

where $G^i : \pi_M^{-1}(U) \to R$ and $\Gamma^i_j : \pi_M^{-1}(U) \to R$ are the local parameters of the semi-spray and the connection, resp. If S is a spray, i.e. homogeneous, then G^i is homogeneous of order 2 in y, thus Γ^j_i must satisfy

$$y^j(\Gamma^i_j \circ \mu_t) = t y^j \Gamma^i_j$$

This relation, however, may be satisfied when Γ^i_j is not homogeneous of order 1 in y as shown in the following example:

$$(n = 2) \qquad (\Gamma^i_1(x, y) = (y_1^3 - y_2)/(y_1^2 + y_2^2), \quad \Gamma^i_2 = (y_2^3 + y_1)/(y_1^2 + y_2^2)$$

This means that a non-homogeneous connection can be compatible to a spray. Thus we have proved the following

Proposition. If a connection structure is compatible to a spray, then the connection is *not* necessarily homogeneous.

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