A QUATERNIONIC TREATMENT OF NAVIER-STOKES EQUATIONS [1)]

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1. Formulation of the problem

The aim of our considerations is to give a quaternionic approach for solving the time-independent NAVIER-STOKES equations:

$$\left.\begin{aligned} -\Delta\hat{u} + \frac{\varrho}{\eta}(\hat{u}\cdot\operatorname{grad})\hat{u} + \frac{1}{\eta}\operatorname{grad} p &= \frac{\varrho}{\eta}\hat{f} \\ \operatorname{div}\hat{u} &= 0 \end{aligned}\right\} \text{ in } G \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

$$\hat{u} = 0 \quad \text{on } \Gamma \;, \qquad (3)$$

where $\hat{u}$ means the velocity of the fluid and p the hydrostatical pressure. Furthermore ϱ denotes the density, η the toughness and $\hat{f}$ the vector of the outer forces. Let G be a bounded domain and Γ its smooth boundary. C.W. OSEEN showed that in the case of a ball approximative solutions of good quality may be obtained if the convection term $C(\hat{u}) = \frac{\varrho}{\eta}(\hat{u}\cdot\operatorname{grad})\,\hat{u}$ is replaced by $\frac{\varrho}{\eta}(\hat{v}\cdot\operatorname{grad})\,\hat{u}$ where $\hat{v}$ denotes the solution of the corresponding STOKES problem. Based on this idea we intend to solve NAVIER-STOKES equations by reduction to a sequence of STOKES problems.

Denote by $1, e_1, e_2, e_3$ the quaternionic units which satisfy the properties

$$e_i e_j + e_j e_i = -2\delta_{ij} \qquad i,j = 1,2,3$$

[1)] This paper is in final form and no version of it will be submitted for publication elsewhere.

Let $\hat{f} = f = \sum_{i=1}^{3} f_i e_i$, $u = u_0 + \sum_{i=1}^{3} u_i e_i$ and $D = \sum_{i=1}^{3} D_i e_i$.
Then the system (1)-(2)-(3) admits the following hypercomplex notation

$$\left.\begin{aligned} D\,D\,u + \frac{\varrho}{\eta} M(u) + \frac{1}{\eta} D\,p &= 0 \\ \mathrm{Re}\, D\,u &= 0 \end{aligned}\right\} \text{ in } G \qquad \begin{matrix}(4)\\(5)\end{matrix}$$

$$u = 0 \quad \text{on } \Gamma \qquad (6)$$

with $M(u) = M^*(u) - \hat{f} = \mathrm{Re}(uD)u - f$, where DIRICHLET's problem

$$\begin{aligned} -\Delta u_0 &= 0 && \text{in } G \\ u_0 &= 0 && \text{on } \Gamma \end{aligned}$$

has been added. Note that $\mathrm{Re}\, u_0 D = 0$. For each $k \in \mathbb{N} \cup \{0\}$ the quaternionic versions of the spaces $W_2^k(G)$ are denoted by $W_{2,H}^k(G)$, where $W_{2,H}^0$ will be identified with $L_{2,H}$.

2. Some preliminary Statements

We may introduce in $L_{2,H}(G)$ considered as a real vector space, the inner product

$$[u,v] = \int_G \bar{u}\, v\, dG \quad \text{with } \bar{u} = u_0 - \sum_{i=1}^{3} u_i e_i \quad . \qquad (7)$$

Obviously $[u,v] \in H$ (skew-field of quaternions). So the values of $[u,v]$ are not necessarily real numbers, but $[u,u] \geq 0$.

Proposition 1 [GS1]

The HILBERT space $L_{2,H}(G)$ admits the orthogonal decomposition

$$L_{2,H}(G) = \ker D \cap L_{2,H}(G) \oplus D\, \mathring{W}_{2,H}^1(G)$$

where $\oplus$ denotes an orthogonal sum according to the inner product (7).

Corollary 1

There exist two orthoprojections $\mathbb{P}$ and $\mathbb{Q}$ with

$$\mathbb{P} : L_{2,H}(G) \xrightarrow{\text{onto}} \ker D \cap L_{2,H}(G) \qquad (8)$$

$$\mathbb{Q} = I - \mathbb{P} : L_{2,H}(G) \xrightarrow{\text{onto}} D\, \overset{\circ}{W}{}^{1}_{2,H}(G) \cap L_{2,H}(G) \qquad (9)$$

Proof.
This is a direct consequence of Proposition 1 . #

Corollary 2
Let $u \in W^1_{2,H}(G)$. Then there holds the differentiation rule

$$D\,\mathbb{Q}\,u = D\,u$$

Proof.
We have $D\mathbb{Q}u = D\,u - D\,\mathbb{P}u$. The definition of the projection $\mathbb{P}$ yields the assertion. #

Now it is necessary to introduce some integral operators. Let be $e(x) = -\frac{1}{4\pi}\sum_{i=1}^{3} x_i |x|^{-3} e_i$. Then we are able to give the following denotations

$$(T_G u)(x) := \int_G e(x-y)u(y)dy$$

$$(F_\Gamma u)(x) := -\int_\Gamma e(x-y)\alpha(y)u(y)d\Gamma_y \ , \ x \notin \Gamma$$

$$(S_\Gamma u)(x) := -2\int_\Gamma e(x-y)\alpha(y)u(y)d\ _y \ , \ x \in \Gamma$$

where $\alpha(y)$ denotes the unit vector of the outer normal at the point y on Γ . $\alpha(y)$ may be written by $\alpha(y) = \sum_{i=1}^{3}\alpha_i e_i$
T_G is the 3-dimensional analogue to the complex T-operator (cf.[V]). F_Γ can be seen as a 3-dimensional analogue to the plane CAUCHY-type integral operator. The operator S_Γ is represented by a singular CAUCHY integral.
For the following it is necessary to put together some essential properties of these operators.

Proposition 2 [GS 1]
1^o Let $1 < p < \infty$, $k = 0,1,\ldots$. Then we have

$$T_G : W^k_{p,H}(G) \longrightarrow W^{k+1}_{p,H}(G) .$$

2^0 Let $1 < p < \infty$, $k = 1,2,\ldots$. Then it holds

$$F_\Gamma : W^{k-1/p}_{p,H}(\Gamma) \longrightarrow W^k_{p,H}(G) \cap \ker D .$$

3^0 The operator $tr\, T_G\, F_\Gamma$ is an isomorphism in the pair of spaces $W^{k-1/2}_{2,H} \cap \mathrm{im} P_\Gamma$, $W^{k+1/2}_{2,H} \cap \mathrm{im} Q_\Gamma$, where $P_\Gamma = \frac{1}{2}(I+S_\Gamma)$ and $Q_\Gamma = \frac{1}{2}(I-S_\Gamma)$. The trace operator tr means the restriction on the boundary Γ.

4^0 The orthoprojections $\mathbb{P}$ and $\mathbb{Q}$ have the algebraic representations

$$\mathbb{P} = F_\Gamma(tr\, T_G F_\Gamma)^{-1} tr\, T_G$$
$$\mathbb{Q} = I - F_\Gamma(tr\, T_G F_\Gamma)^{-1} tr T_G$$

5^0 The operators $\mathbb{P}$ and $\mathbb{Q}$ are acting within the space $W^k_{p,H}(G)$, $1 < p < \infty$, $k = 1,2,\ldots$.

6^0 It is clear that $\|\mathbb{P}\|_{L(L_{2,H})} = \|\mathbb{Q}\|_{L(L_{2,H})} = 1$.

3. About the Solution of NAVIER-STOKES' Equations

By the aid of the introduced operators the problem (4)-(5)-(6) admits a certain advandageous form.

Proposition 3 [G]

Let $f \in L_{2,H}(G)$, $p \in W^1_2(G)$. Every solution of system (4)-(5)-(6) may be represented by

$$u = -\frac{\varrho}{\eta} T_G \mathbb{Q} T_G M(u) - \frac{1}{\eta} T_G \mathbb{Q} p \qquad (10)$$

$$\mathrm{Re}\, \frac{\varrho}{\eta} \mathbb{Q}\, T_G M(u) + \frac{1}{\eta} \mathrm{Re}\, \mathbb{Q}\, p = 0 . \qquad (11)$$

Now it arises the question if the systems (10)-(11) and (4)-

(5)-(6) are equivalent. We have the following result:

THEOREM 1 (Equivalence)
Let $u \in \overset{\circ}{W}{}^1_{2,H}(G)$, $p \in L_2(G)$ a solution of system (10)-(11). Then $\hat{u} = \operatorname{Im} u$ is a weak solution of system (4)-(5)-(6). Conversely, if u is a weak solution of system (4)-(5)-(6) then there exists a function $p \in L_2(G)$ such that the pair $\{u,p\}$ with $u = \hat{u}$ solves system (10)-(11)

Proof.
For the proof we note that a weak solution of the system (4)-(5)-(6) is given if there is fulfilled the following identity

$$\frac{\eta}{\varrho}\sum_{i=1}^{3}(\operatorname{grad} u_i,\ \operatorname{grad} v_i) + \sum_{i=1}^{3}(u_i D_i u, v) = (f,v)$$

and $v \in \ker \operatorname{div} \cap \overset{\circ}{W}{}^1_{2,H}(G)$. With (u,v) we denote the product $(u,v) = \sum_{i=1}^{3}\int_G u_i v_i \, dG$. By the help of partial integration a straightforward computation we get the wanted result. #

THEOREM 2 ("Almost"-a-priori Estimate)
Let $\{u,p\} \in \overset{\circ}{W}{}^1_{2,H}(G) \cap \ker \operatorname{div} \times L_2(G)$ be a solution of system (10)-(11). The inequality

$$\frac{\lambda_1}{1+\lambda_1}^{1/2}\|U\|_{W^1_{2,H}} + \frac{1}{\eta}\|Q\,p\|_{L_{2,H}} \leq 2^{1/2}\frac{\varrho}{\eta}\|T_G M(u)\|_{L_{2,H}} \quad (12)$$

is valid. λ_1 denotes the smallest eigenvalue of the problem $\{-\Delta u = \lambda u,\ \operatorname{tr} u = 0\}$.

Proof.
Representation (10) leads to

$$D\,u + \frac{1}{\eta}Q\,p = \frac{\varrho}{\eta}Q\,T_G M(u) \quad (13)$$

It is clear that $u \in \overset{\circ}{W}{}^1_{2,H}(G)$, $Du \in \operatorname{im} \mathbb{Q}$, $\operatorname{Re} Du = 0$ and $\operatorname{Im} p = 0$. Thence it follows

$$\operatorname{Re} [Du, \mathbb{Q} p] = \operatorname{Re} [Du, p] - \operatorname{Re} [Du, \mathbb{P} p] = 0 \qquad (14)$$

as $Du \in \operatorname{im} \mathbb{Q}$ and $\operatorname{Re} Du = 0$. From (13) and (14) we deduce the identity

$$\|Du\|^2_{L_{2,H}} + \frac{1}{\eta^2}\|\mathbb{Q} p\|^2_{L_{2,H}} = \frac{\varrho^2}{\eta^2} \|\mathbb{Q} T_G M(u)\|^2_{L_{2,H}}$$

and therefore

$$2^{-1/2} (\|Du\|_{L_{2,H}} + \frac{1}{\eta} \|\mathbb{Q} p\|_{L_{2,H}}) \leq \frac{\varrho}{\eta} \| T_G M(u)\|_{L_{2,H}} .$$

Using POINCARE's inequality it is possible to obtain that $\|Du\|_{L_{2,H}} \geq c^{-1} \| u\|_{L_{2,H}}$, where $c = \inf_{i=1,2,3;\, P_G \supset G} c^{(i)}(G)$ and P_G is a right parallelepiped with the length of the edges $c^{(i)}(G)$, $i=1,2,3$. Together with Corollary 4.2 in [GS2] we gain

$$\| Du\|_{L_{2,H}} \geq \left(\frac{\lambda_1}{1 + \lambda_1}\right)^{1/2} \| u\|_{W^1_{2,H}}$$

and so our statement. #

Remark
Inequality (12) has the same structure as the a-priori estimate for STOKES' equations. Indeed, it is not an a-priori estimate for the solutions of NAVIER-STOKES equations, but it is without doubt very important with respect to further considerations. Making use of (12) one may estimate the term of the hydrostatical pressure $\mathbb{Q} p$ by the velocity u and the right-hand side f.

Proposition 4 [G]
Let $u \in \overset{\circ}{W}{}^1_{2,H}(G)$ $1 < p < 3/2$. Then

$$\|M^*(u)\|_{L_{p,H}} \leqslant C_1 \|u\|^2_{\overset{\circ}{W}{}^1_{2,H}}$$

*

Remark

The constant C_1 may be estimated by the inequality

$$C_1 \leqslant 9^{1/p} \|T_G\|_{[L_{2,H},L_{q,H}]}$$

where $q<6$ and $p = \frac{2q}{2+q}$. Using results in [Sm] one can get for any $q<6$ the estimate

$$\|T_G\|_{[L_{2,H},L_{q,H}]} \leqslant (\text{diam } G)^{-\frac{1}{2}+\frac{3}{q}} \, 2^{\frac{7}{2}+\frac{2}{q}} \pi^{\frac{1}{2}+\frac{1}{q}} \left(\frac{5}{2}\right)^{-1/2-1/q} q^{1/q} .$$

THEOREM 3 (Existence and Iteration procedure)

System (10)-(11) has a unique solution $\{u,p\} \in \overset{\circ}{W}{}^1_{2,H}(G) \cap$ $\cap \ker\operatorname{div} \times L_2(G)$ (p is unique up to a real constant) if the right-hand side f satisfies the condition

$$\frac{\varrho}{\eta} \|f\|_{L_{p,H}} \leqslant (16k^2C_1)^{-1}$$

with $$K = \frac{\varrho}{\eta} \|T_G\|_{[L_{2,H}\cap \operatorname{im}\mathbb{Q}, \overset{\circ}{W}{}^1_{2,H}]} \|T_G\|_{[L_{p,H},L_{2,H}]} .$$

For any function $u_0 \in \overset{\circ}{W}{}^1_{2,H}(G) \cap \ker\operatorname{div}$ with

$$\|u_0\| \leqslant \min(R, (1/4KC_1)+W)$$

$$(R = (2KC_1)^{-1},\ W = \left[(4KC_1)^{-2} - \varrho\|f\|_{L_{p,H}}/(\eta C_1)\right]^{1/2})$$

the iteration method

$$u_n = -\frac{\varrho}{\eta} T_G\mathbb{Q}\, T_G\, M(u_{n-1}) - \frac{1}{\eta} T_G\, \mathbb{Q}\, p_n \qquad n=1,2,\ldots \qquad (15)$$

$$\frac{1}{\eta}\operatorname{Re} Q p_n = -\frac{\varrho}{\eta}\operatorname{Re} Q T_G M(u_{n-1}) \tag{16}$$

converges in $\overset{\circ}{W}{}^1_{2,H}(G) \times L_2(G)$

Proof.
For the proof we remark that we reduce the NAVIER-STOKES problem in accordance with (15)-(16) to a sequence of STOKES' problems. Several estimates yield the essential inequality $\|u_n\|_{W^1_{2,H}} \leq \|u_{n-1}\|_{W^1_{2,H}}$. BANACH's fixed-point theorem finishes the proof. A detailed discussion of the proof is given in [G]. #

Corollary
Under the suppositions of Theorem 3 we have

(i) $\|u\|_{W^1_{2,H}} \leq (4KC_1)^{-1}-W$

(ii) $\|u_n-u\|_{W^1_{2,H}} \leq L^n\left[(4KC_1)^{-1}-W\right]$,

where $L = 1 - 4KC_1W < 1$. #

THEOREM 4 (Regularity)
Let $f \in W^k_{q,H}(G)$ $q > 6/5$. Then the solution $\{u,p\}$ of the system (10)-(11) belongs to $W^{k+2}_{q,H}(G) \cap \overset{\circ}{W}{}^1_{2,H}(G) \times W^{k+1}_q(G)$.

Proof.
We confine our considerations to the case $f \in L_{q,H}(G)$. In the general case the proof is practicable by the same technique. First we consider the STOKES problem

$$v + \frac{1}{\eta} T_G Q g = \frac{\varrho}{\eta} T_G Q M(u) \tag{17}$$

$$\frac{1}{\eta}\operatorname{Re} Q g = -\frac{\varrho}{\eta}\operatorname{Re} Q T_G M(u) \; . \tag{18}$$

Using Theorem 3 and the representation of the solution of

STOKES' problem (cf. [GS 1]) we get $v = u \in W^2_{s,H}(G)$ and $g = p \in W^1_s(G)$, $s < 3/2$. By help of HOELDER's inequality and embedding theorems we gain $M^*(u) \in L_{t,H}(G)$ for all $t < 3$. Now let $q < 3$, $M(u) \in L_{t,H}(G)$.
Then, by analysis of problem (17)-(18) we find $u \in W^2_{t,H}(G)$ and $p \in W^1_t(G)$. Renewing this procedure over and over again we may achieve $M^*(u) \in L_{r,H}(G)$ for $r < \infty$. So it follows $M(u) \in L_{q,H}(G)$. A new reflection of STOKES problem (17)-(18) leads to the wanted result. #

Remark
The hypercomplex investigation of NAVIER-STOKES problem has following advantages:

1° With a unified method may be solved all essential analytical problems as existence, uniqueness, regularity and "almost"-a-priori estimate.

2° Approximative methods may be chosen within the same calculus.

3° It is not necessary to use monotony principles.

4° Approximative solutions u_n for the exact solution u in $W^1_{2,H}$ strongly converge. Most of the other methods only deliver a weak convergence or the strong convergence of a subsequence.

5° There are good possibilities for the judgement of the quality of the approximative solutions.

6° Our procedure enables us to connect the computations with a suitable boundary collocation method.

4. Numerical Solution of Boundary Value Problems of NAVIER-STOKES Equations

In this section we intend to demonstrate the numerical solution of elliptical boundary value problems by the help of a discrete function theory. In this connection we shall deal with boundary value problems of STOKES' equations and then consider an iteration method of solving NAVIER-STOKES equations.

Elements of a discrete generalized function theory were developed in the paper [GS3]. We will use the same denotations here and that is why we define only some essential subjects. We introduce the equidistant lattice

$R^3_h=\{(ih,jh,kh),\ i,j,k \text{ integer},\ h>0 \text{ real}\}$ and $G_h=G\cap R^3_h$.

The translation of $x\in R^3_h$ by $\pm h$ in the x_i-direction shall be denoted by $V^{\pm}_{i,h}x$. Then we can define generalized discrete CAUCHY-RIEMANN operators by

$$(D^{\pm}_h f)(x)=\pm\sum_{i=1}^{3} e_i[f(V^{\pm}_{i,h}x)-f(x)]/h=\sum_{i=1}^{3} e_i(D^{\pm}_{i,h}f)(x)$$

and a discretization of the Laplacian by

$$-\Delta_h=D^+_hD^-_h=D^-_hD^+_h\ .$$

If we denote by E_h the fundamental solution of $\{-\Delta_h,\mathrm{tr}\}$ (constructed in [GS1,GS3]), then $e^{\pm}_h=D^{\mp}_hE_h$ are fundamental solutions of D^+_h and D^-_h, respectively. Additionally define some denotations

$\partial G_h = \{x\in G_h:\mathrm{dist}(x,\mathrm{co}\ G_h)\le 3^{1/2}h\}$

$\partial G_{h,l} = \{x\in \partial G_h:\exists i\in\{1,2,3\},\ \text{with}\quad V^-_{i,h}x\notin G_h\}$

$\partial G_{h,r} = \{x\in \partial G_h:\exists i\in\{1,2,3\},\ \text{with}\quad V^+_{i,h}x\notin G_h\}$

$\partial G_{h,l,i} = \{x\in \partial G_h:V^-_{i,h}x\notin G_h\}\ ,\ i=1,2,3$

$\partial G_{h,r,i} = \{x\in \partial G_h:V^+_{i,h}x\notin G_h\}\ ,\ i=1,2,3$

$\partial G_{h,l,i,j} = \partial G_{h,l,i}\cap \partial G_{h,l,j}\qquad i,j\in\{1,2,3\}$

$\partial G_{h,l,i,j,k} = \partial G_{h,l,i}\cap \partial G_{h,l,j}\cap \partial G_{h,l,k}\qquad i,j,k\in\{1,2,3\}$

and introduce discrete analoga to the operator T_G by

$$(T^+_hf)(x):=\sum_{y\in \mathring{G}_h\cup\partial G_{h,l}} e^+_h(x-y)f(y)h^3-\sum_{\substack{k,j=1\\ j>k}}^{3}\ \sum_{y\in\partial G_{h,l,j,k}} e^+_h(x-y)f(y)h^3+$$

$$+\sum_{\substack{y\in\partial G_{h,l,i,j,k}\\ i\neq j\neq k}} e^+_h(x-y)f(y)h^3$$

$$(T_h^- f)(x) := \sum_{y \in \mathring{G}_h \cup \partial G_{h,r}} e_h^-(x-y)f(y)h^3 - \sum_{\substack{k,j=1 \\ j>k}} \sum_{y \in \partial G_{h,r,j,k}} e_h^-(x-y)f(y)h^3 + + \sum_{y \in \partial G_{h,r,i,j,k}} e_h^-(x-y)f(y)h^3$$

Further we define the quaternionic-valued inner product

$$\langle f,g \rangle = \sum_{x \in G_h} \overline{f(x)}\, g(x)h^3$$

and $L_{2,h}(G_h)$ by the help of the induced norm $\langle f,f \rangle$. $L_{2,h}(G_h)$ allows the orthogonal decomposition

$$L_{2,h}(G_h) = \ker D_h^+(\text{int } G_h) \oplus D_h^-(\mathring{W}_{2,h}^{1,-}(G_h)) \quad \text{(see [GS3])}.$$

If we introduce an analogue to the operator F_Γ by

$$F_h^+ f = f - T_h^+ D_h^+ f$$

then we have a discrete version of BOREL-POMPEIU's formula and it can be shown that $F_h^+ f$ is uniquely determined by the boundary values of f. The orthoprojections onto $\ker D_h^+(\text{int } G_h)$ and $D_h^-(\mathring{W}_{2,h}^{1,-}(G_h))$, respectively, are given in the following manner:

$$P_h^+ = F_h^+(\text{tr } T_h^- F_h^+)^{-1} \text{tr } T_h^-, \quad Q_h^+ = I - P_h^+ .$$

A straightforward computation shows the following properties

$$D_h^{\pm} T_h^{\pm} = I, \quad D_h^{\pm} Q_h^{\pm} = D_h^{\pm}.$$

Some further results shall be given here without proof.

<u>Lemma 1:</u> [GS1]

For $1<p<3$ and $q<\frac{3p}{3-p}$ the operators

$$T_h : L_{p,h,H}(G_h) \longrightarrow L_{q,h,H}(G_h)$$

are continuous. *

<u>Lemma 2:</u>

Let $u \in \mathring{W}_{2,h,H}^{1,-}(G_h)$ and $q<6$. Then it holds

$$\|u\|_{L_{q,h,H}(G_h)} \leq C\|u\|_{W_{2,h}^1}.$$

<u>Proof.</u>

It is clear that $D_h^- u \in L_{2,h,H}$, whence follows

$$\|u\|_{q,h,H} \leq \|T_h^- D_h^- u\|_{q,h,H} \leq C\|D_h^- u\|_{2,h,H} \leq C\|u\|_{W_{2,h}^1}.$$

In this case we made use of the discrete BOREL-POMPEIU formula and Lemma 1. *

In the discrete case it is also possible to deduce a relation between the smallest eigenvalue $\lambda_{1,h}(G_h)$ of $\{-\Delta_h, \text{tr}\}$ and the norm of T_h^-. A simple calculation yields

<u>Lemma 3:</u>

$$\|T_h^- f\|_{2,h,H} \leq (\lambda_{1,h}(G_h))^{-1/2} \|f\|_{2,h,H} \qquad \forall f \in \text{im } Q_h^+$$

$$\|T_h^- f\|_{2,1,h,H} \le (1+\lambda_{1,h}(G_h))^{1/2}\|f\|_{2,h,H} \quad \forall f \in \text{im } Q_h^+ .$$

Proposition 5

Let $f,g \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h)$. Then $\langle D_h^+ f,g\rangle = \langle f,D_h^- g\rangle$ #

Remark

The functions f and g may be extended by zero into the domain co G_h. Therefore the definition of $D_h^+ f$ and $D_h^- f$ does not cause any difficulties in points on ∂G_h.

Proposition 6

Let $f,g \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h)$. Then $\text{Re}\langle D_h^+ \text{Re } f,g\rangle=\text{Re}\langle f,\text{Re } D_h^- g\rangle$. # (19)

In the following we shall briefly write $\text{Re}\langle\cdot,\cdot\rangle=:[\cdot,\cdot]_h$. Now we define

$$\text{grad}_h^+ f:=(D_{1,h}^+,D_{2,h}^+,D_{3,h}^+)^T , \qquad \text{div}_h^- v:=\sum_{i=1}^{3} D_{i,h}^- v_i .$$

Identity (19) may be written in the form

$$[\text{grad}_h^+ u,v]_h= [u,-\text{div}_h^- v]_h,$$

where $u,v \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h)$, $u:G \longrightarrow R^1$, $v:G \longrightarrow R^3$. Thence

$$(\text{grad}_h^+)^* = -\text{div}_h^-$$

and consequently we obtain

$$\ker \text{div}_h^- = (\text{im grad}_h^+)^\perp ,$$

where the orthogonality is to be understood with respect to the scalar product (19).

Proposition 7

Let $u \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h) \cap \ker \text{div}_h^-$, $p \in L_{2,h,R}(G_h)$ with $\text{Im } p=0$. There is valid $[D_h^- u,Q_h^+ p]_h= 0$.

Proof.

Because of $D_h^- u \in \text{im } Q_h^+$, $\text{Re } D_h^- u=0$ and $\text{Im } p=0$ we get

$$[D_h^- u,Q_h^+ p]_h=[Q_h^+ D_h^- u,p]_h=[D_h^- u,p]_h=0. \qquad \#$$

THEOREM 5

For each $f \in L_{2,h,H}$ there exist H-valued functions $u \in \overset{\circ}{W}{}^{1,-}_{2,h}(G_h) \cap \ker \text{div}_h^-$ and $p \in L_{2,h,R}$ with $\text{Im } p=0$ such that

$$\frac{\varrho}{\eta} Q_h^+ T_h^+ f = D_h^- u + \frac{1}{\eta} Q_h^+ p . \qquad (20)$$

Proof.

Obviously hold $D_h^- u \in \operatorname{im} Q_h^+$, $Q_h^+ p \in \operatorname{im} Q_h^+$. With respect to the validity of Proposition 7 it is necessary to show that the relations

$$[D_h^- u, Q_h^+ T_h^+ f]_h = 0 \quad \text{for} \quad u \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h) \cap \ker \operatorname{div}_h^-$$

$$[Q_h^+ p, Q_h^+ T_h^+ f]_h = 0 \quad \text{for} \quad p \in L_{2,h,R} \quad \text{with} \quad \operatorname{Im} p = 0$$

imply $f=0$. First we obtain

$[D_h^- u, Q_h^+ T_h^+ f]_h = [D_h^- u, T_h^+ f]_h = [u,f]_h = 0$ and therefore $f = \operatorname{grad}_h^+ q$ with $\operatorname{Im} q = 0$. On the other hand, we have $\|Q_h^+ q\|_{2,h,H} = 0$, whence $\operatorname{grad}_h^+ q = D_h^+ Q_h^+ q = 0$ and $f=0$. #

The operator T_h^- applying to representation (20) then follows, by making use of the discrete BOREL-POMPEIU formula, the existence of a decomposition of the function $T_h^- Q_h^+ T_h^+ f$ into the sum

$$u + \frac{1}{\eta} T_h^- Q_h^+ p = \frac{\varrho}{\eta} T_h^- Q_h^+ T_h^+ f$$

with suitably chosen discrete functions

$$u \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h) \cap \ker \operatorname{div}_h^-, \quad p \in L_{2,h,R}, \ \ker \operatorname{Im} p = 0 .$$

Summarizing we may formulate for the solution of the discrete STOKES' problem.

THEOREM 6

The discrete boundary value problem

$$-\Delta_h u + \frac{1}{\eta} \operatorname{grad}_h^+ p = \frac{\varrho}{\eta} f \quad \text{in int } G_h \tag{21}$$

$$\operatorname{div}_h^- u = 0 \quad \text{in int } G_h \tag{22}$$

$$u = 0 \quad \text{on } \partial G_h \tag{23}$$

has for every $f \in L_{2,h,H}$ a solution $\{u,p\}$, where u and p are uniquely defined (p up to a real constant !).

Proof.

The existence has already been shown. Formula (20) and Proposition 7 yield

$$\|Q_h^+ T_h^+ f\|^2_{2,h,H} = \|D_h^- u\|^2_{2,h,H} + \frac{1}{\eta^2} \|Q_h^+ p\|^2_{2,h,H}$$

whence

$$\|D_h^- u\|_{2,h,H} + \frac{1}{\eta} \|Q_h^+ p\|_{2,h,H} \le 2^{1/2} \|Q_h^+ T_h^+ f\|_{2,h,H} \tag{24}$$

This a-priori estimate leads us to the uniqueness of u. Assuming the existence of two solutions (u,p_1) and (u,p_2), thus it immediately implies $p_1-p_2 \in \ker Q_h^+$, therefore $p_1-p_2 \in \ker D_h^+(\mathrm{int}\, G_h)$ and $p_1-p_2=\mathrm{const.} \in R^1$. #

<u>Corollary 4</u>

There is valid the a-priori estimate

$$(\lambda_{1,h}/(1+\lambda_{1,h}))^{1/2}\|u\|_{2,1,h,H}+\frac{1}{\eta}\|Q_h^+p\|_{2,h,H}\le 2^{1/2}\|T_h^+f\|_{2,h,H}\cdot\frac{\varrho}{\eta}$$

<u>Proof.</u>

It may be proved by using Lemma 2 and Lemma 3. #

<u>Remark</u>

The treatment of the discrete STOKES problem points out a wide correspondence with the continuous case. This relates to both the method of consideration and the concrete formulation of the results.

<u>Remark</u>

The discrete boundary value problem (21)-(22)-(23) may be interpreted by a scheme of finite differences. That means the presented application of discrete function theory can be seen as a new approach to the construction and analytical investigation of finite difference methods.

In a simple way a-priori estimates, for instance (24) allow the investigation of stability problems.

Now we shall deal with DIRICHLET's problem for NAVIER-STOKES equations. Considering the latter formulated results for the STOKES problem, the non-linear term $M^*(u)=\frac{\varrho}{\eta}(u,\mathrm{grad})u$ remains to be discretisized in a proper way.

Define

$$M_h^{*,-}(u):=\frac{\varrho}{\eta}(u,\mathrm{grad}_h^-)u \quad \text{and} \quad M_h^-(u):=M_h^{*,-}(u)-\frac{\varrho}{\eta}f \quad \text{so it}$$

holds

$$\|\tfrac{\eta}{\varrho}M_h^{*,-}(u)\|_{p,h,H}^p\le\|\sum_{i,j=1}^{3}u_iD_{i,h}^-u_je_j\|_{p,h,H}^p\le\sum_{i,j=1}^{3}\|u_iD_{i,h}^-u_j\|_{p,h,H}=$$

$$=\sum_{i,j=1}^{3}\sum_{y\in G_h}|u_i(y)D_{i,h}^-u_j(y)|^p h^3\le\sum_{i,j=1}^{3}\|u_i\|_{q,h,H}^p\|D_{i,h}^-u_j\|_{2,h,H}^p\le$$

$$\le\sum_{i,j=1}^{3}C^p\|u_i\|_{2,1,h,H}^p\|u_j\|_{2,1,h,H}^p\le 9C^p\|u\|_{2,1,h,H}^{2p}$$

with $q<6$, $p=\frac{2q}{2+q}$. To obtain this estimate, we used Lemma 2 and HOELDER's inequality for sums. Consider the boundary value problem

$$-\Delta_h u + \frac{1}{\eta}\mathrm{grad}_h^+ p + \frac{\varrho}{\eta}(u,\mathrm{grad}_h^-)u = \frac{\varrho}{\eta}f \quad \text{in int } G_h \tag{25}$$

$$\mathrm{div}_h^- u = 0 \quad \text{in int } G_h \tag{26}$$

$$u = 0 \quad \text{on } \partial G_h \tag{27}$$

Applying the discrete generalized VEKUA's theory we obtain the following equivalent problem

$$u = -T_h^- Q_h^+ T_h^+ M_h^-(u) - \frac{1}{\eta} T_h^- Q_h^+ p \quad \text{in } G_h \tag{28}$$

$$\mathrm{Re}\ Q_h^+ T_h^+ M_h^-(u) = \frac{1}{\eta}\ \mathrm{Re}\ Q_h^+ p \quad \text{in } G_h \tag{29}$$

Consider the following iteration procedure:

$$u_n = -T_h^- Q_h^+ T_h^+ M_h^-(u_{n-1}) - \frac{1}{\eta} T_h^- Q_h^+ p_n \tag{30}$$

$$\mathrm{Re}\ Q_h^+ T_h^+ M_h^-(u_{n-1}) = \frac{1}{\eta}\ \mathrm{Re}\ Q_h^+ p_n \tag{31}$$

$$u_o \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h) \cap \ker \mathrm{div}_h^-, \quad n=1,2,3,\ldots$$

It is known from Theorem 6 that the STOKES problems (30)-(31) have a solution in each case. Therefore the iteration procedure may be carried out. Since the norms of T_h^-, T_h^+, Q_h^+ and M_h^- can be estimated in a similar manner as in the continuous case, the whole proof of convergence of iteration is to be carried out analogously. #

<u>THEOREM 7</u>

System (30)-(31) has a unique solution $\{u,p\}$, where $u \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h) \cap \ker \mathrm{div}_h^-$, $p \in L_{2,h,R}$ (p is uniquely defined up to a real constant) if

$$\frac{\varrho}{\eta}\|f\|_{p,h,H} \le (16K_h^2 C_{1,h})^{-1} \tag{32}$$

For every function $u_o \in \overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h) \cap \ker \mathrm{div}_h^-$ with

$$\|u_o\|_{2,1,h,H} \le R_h \tag{33}$$

the procedure (30)-(31) converges in $\overset{\circ}{W}{}^{1,-}_{2,h}(G_h) X L_{2,h,H}$ to the solution of the problem (28)-(29).

<u>Proof.</u>

For the proof we refer to the continuous case. #

<u>Remark</u>

There hold the following relations for the constants used.

$$K_h = \|T_h^-\|_{[L_{2,h,H} \cap \mathrm{im}\, Q_h^+, \overset{\circ}{W}{}^1_{2,h,H}]} \|T_h^+\|_{[L_{p,h,H}, L_{2,h,H}]}$$

$C_{1,h} = 9^{1/p} C_h \frac{\varrho}{\eta}$

$W_h = \{(4K_h C_{1,h})^{-2} - \varrho \|f\|_{p,h,H} (\eta C_{1,h})^{-1}\}^{1/2}$

$R_h = (4K_h C_{1,h})^{-1}$

An analysis of the proof of Lemma 1 shows that the norms $\|T_h^-\|_{[L_{p,h,H}, L_{2,h,H}]}$ and $\|T_h^-\|_{[L_{2,h,H}, L_{q,h,H}]}$ (similarly $\|T_h^+\|$) can be uniformly estimated with respect to h. In this way the embedding constant $C_{1,h}$ is also to be bounded uniformly. Using the monotony property of the eigenvalues $\lambda_{1,h}(G_h)$ in case of the embedding in a larger domain (for instance, in a described cube), we finally can find a uniform estimate for K_h and later also for R_h.
If f is RIEMANN-integrable, then $\|f\|_{L_{p,h,H}(G)}$ converges such that W_h is uniformly bounded. These facts allow to formulate Theorem 7 in such a way that the right-hand sides contain only terms which do not depend on h. The exact formulation will be omitted here.

Corollary 5

(i) There holds

$\|u\|_{2,1,h,H} \leq (4K_h C_{1,h})^{-1} - W_h$

(ii) Let $L_h = (4K_h C_{1,h})^{-1} - 4K_h C_{1,h} W$. Then we have

$\|u_n - u\|_{2,1,h,H} \leq L_h^n \|u_o - u\|_{2,1,h,H}$

If $u_o = \emptyset$, then is valid

$\|u_n - u\|_{2,1,h,H} \leq L_h^n \{(4K_h C_{1,h})^{-1} - W_h\}$ #

Now we can finish our considerations of the discrete boundary value problem for NAVIER-STOKES equations.
For every $h > \emptyset$ the question of existence and uniqueness was clarified, an a-priori estimate of the solution could be given, and the speed of the convergence was defined. The fixed-point priciple ensures the stability of the introduced iteration procedure. For the discrete STOKES problems which are to be solved in each step of the iteration method we could prove the unique solvability. For all constants which occurred explicit bounds could be found. Now we shall turn to

the numerical realization of the proposed method.

THEOREM 8 [GS1]

Set $v_h = T_h^- Q_h^+ T_h^+ M_h^-(u) + u + \frac{1}{\eta} T_h^- Q_h^+ p$.

If $f \in L_{p,H}(G)$ then

$$v_h \longrightarrow 0 \quad \text{for} \quad h \longrightarrow 0.$$

#

Our next aim is to find an error estimate in the space $\overset{\circ}{W}{}^{1,-}_{2,h,H}(G_h)$. We prove the following result.

THEOREM 9

Let $f \in \mathcal{R} \cap L_\infty$. Then there is valid

$$\|u - u_h\|_{W^1_{2,h,H}(G_h)} \longrightarrow 0 \quad \text{for} \quad h \longrightarrow 0.$$

Proof.

Theorem 8 and Theorem 3 yield

$$D_h^-(u-u_h) + \frac{1}{\eta} Q_h^+(p-p_h) = Q_h^+ T_h^+ (M_h^-(u) - M_h^-(u_h)) + w_h \quad (34)$$

Acting D_h^+ on (34) it follows $\operatorname{Re} D_h^+ w = 0$. Furthermore we have

$$w_h = Q_h^+ T_h^+ D_h^+ w_h = Q_h^+ T_h^+ g_h .$$

Denote $M_h^-(u) - M_h^-(u_h) + g_h$ with f_h, then (34) can be given in the form

$$D_h^-(u-u_h) + \frac{1}{\eta} Q_h^+(p-p_h) = Q_h^+ T_h^+ f_h$$

Therewith $(u-u_h, p-p_h)$ are solutions of a discrete STOKES problem. Proposition 7 and Theorem 5 yield the orthogonality of $D_h^-(u-u_h)$ and $\frac{1}{\eta} Q_h^+(p-p_h)$.

Scalar multiplication of this equation with $Q_h^+(p-p_h)$ leads to

$$\frac{1}{\eta}\|Q_h^+(p-p_h)\|^2_{2,h,H} = -[D_h^-(u-u_h), Q_h^+(p-p_h)]_h +$$

$$+ [Q_h^+ T_h^+ (M_h^-(u) - M_h^-(u_h)), Q_h^+(p-p_h)]_h + [w_h, Q_h^+(p-p_h)]_h \leq$$

$$\leq \{\|Q_h^+ T_h^+ (M_h^-(u) - M_h^-(u_h))\|_{2,h,H} + \|w_h\|_{2,h,H}\} \|Q_h^+(p-p_h)\|_{2,h,H}$$

and therefore

$$\frac{1}{\eta}\|Q_h^+(p-p_h)\|_{2,h,H} \leq \|Q_h^+ T_h^+ (M_h^-(u) - M_h^-(u_h))\|_{2,h,H} + \|w_h\|_{2,h,H} .$$

The identity

$$u - u_h = -TQTM(u) - \frac{1}{\eta} TQp + T_h^- Q_h^+ T_h^+ M_h^-(u_h) + \frac{1}{\eta} T_h^- Q_h^+ p_h$$

yields

$$\|u-u_h\|_{2,1,h,H} \le \|TQTM(u)-T_h^-Q_h^+T_h^+M_h^-(u) + \frac{1}{\eta}(TQp-T_h^-Q_h^+p)\|_{2,1,h,H} +$$

$$+ \|T_h^-Q_h^+T_h^+(M_h^-(u)-M_h^-(u_h))\|_{2,1,h,H} + \frac{1}{\eta}\|T_h^-Q_h^+(p-p_h)\|_{2,1,h,H} \le$$

$$\le \|v_h\|_{2,1,h,H} + \|T_h^-Q_h^+T_h^+(M_h^-(u)-M_h^-(u_h))\|_{2,1,h,H} +$$

$$+ \frac{1}{\eta}(1+\lambda_{1,h}^{-1})^{1/2}\|Q_h^+(p-p_h)\|_{2,h,H} \le$$

$$\le \|v_h\|_{2,1,h,H} + 2(1+\lambda_{1,h}^{-1})^{1/2}\|Q_h^+T_h^+(M_h^-(u)-M_h^-(u_h))\|_{2,h,H} +$$

$$+ (1+\lambda_{1,h}^{-1})^{1/2}\|w_h\|_{2,h,H} \le$$

$$\le 2K_h\|w_h\|_{2,h,H} +$$

$$+ 2K_hC_{1,h}\|u-u_h\|_{2,1,h,H}(\|u\|_{2,1,h,H}+\|u_h\|_{2,1,h,H})$$

With the condition $\frac{6}{5}<p<\frac{3}{2}$ immediately follows

$$\|u-u_h\|_{2,1,h,H} \le$$

$$\le 2K_h\|w_h\|_{2,h,H}\{1-2K_hC_{1,h}(\|u\|_{2,1,h,H}+\|u_h\|_{2,1,h,H})\}^{-1}$$

Supposing (32) and (33) we have

$$\|u_h\|_{2,1,h,H} \le (4K_hC_{1,h})^{-1} - W_h \tag{35}$$

and for u Corollary 3 yields

$$\|u\|_{2,1,G} \le (4KC_1)^{-1} - W .$$

These two inequalities, the possibility of a uniform estimate of K_h and $C_{1,h}$ and the RIEMANN-integrability of u and D_iu $(i=1,2,3)$ ensure that for sufficiently small h it uniformly holds with respect to h

$$1-2K_hC_{1,h}(\|u\|_{2,1,h,H}+\|u_h\|_{2,1,h,H}) > c > 0$$

Now we can describe the convergence $u_h \longrightarrow u$ in dependence on the properties of w_h which we have already considered. #

REFERENCES

[G]: GÜRLEBECK,K.: Approximate solution of stationary Navier-Stokes equations, Math. Nachr., to appear

[GS1]: GÜRLEBECK,K.;SPRÖSSIG,W.: Quaternionic Analysis and Elliptic Boundary Value Problems, Akademie-Verlag Berlin, to appear

[GS2]: GÜRLEBECK,K.;SPRÖSSIG,W.: A Unified Approach to Estimation of Lower Bounds for the First Eigenvalue of Several Elliptic Boundary Value Problems, Math. Nachr. 131, (1987), 183-199

[GS3]: GÜRLEBECK,K.;SPRÖSSIG,W.: A Generalized Leibniz Rule and a Discrete Quaternionic Analysis, Suppl. Rend. Circ. Mat. Palermo, Serie II, num. 16, (1987), 44-64

[Sm]: SMIRNOV,W.I.: Lehrgang der höheren Mathematik, Teil V, DVW, Berlin, (1967)

[V]: VEKUA,I.N.: Generalized analytic functions, Reading 1962

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