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VECTOR FIELDS AND CONNECTION ON FIBRED MANIFOLDS *

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If is known, see [1] , [2] , [3] , that every differential equation of second order on a manifold M determines connections on TM. In [3] we have established the set C_{Γ}^{∞} TM of such vector fields on TM by which it is possible to constructe connections on TM, we have found all natural differential operators of first order from C_{Γ}^{∞} TM into the space of all connections on TM. In this paper we generalise some of these constructions in the case of vector fields on fibred manifolds. All manifolds and maps are assumed to be smooth.

1. Tangent value 1-forms and connections on fibre manifolds.

Let $\pi: Y \to M$ be a fibred manifold. A TY-value 1-form ω on Y will be called fibred if $\omega(VY) \subset VY$. If (x^i, y^α) is a chart on Y then expression of a fibred 1-form is

$$\omega = a^i_j(x,y)dx^j \otimes \partial/\partial x^i + (a^\alpha_i(x,y)dx^i + a^\alpha_\beta(x,y)dy^\beta) \otimes \partial/\partial y^\alpha.$$

Let $\pi_1: Z \to M$ be another fibred manifold. Denote by $\pi^* Z$ the π-pull-back of Z, $\pi^* Z = Y x_M Z$. Every fibred TY-valued form ω determines the forms $\omega_h: Y \to \pi^*(TM \otimes T^*M)$ and $\omega_v: Y \to \to VY \otimes V^*Y$, where $\omega_v(X) = \omega(X)$, $X \in VY$ and $\omega_h(X) = T\pi \cdot \omega(U)$ for $T\pi(U) = X$. In coordinates $\omega_h = a^i_j dx^j \otimes \partial/\partial x^i$, $\omega_v = = a^\alpha_\beta dy^\beta \otimes \partial/\partial y^\alpha$.

A connection Γ on Y can also be viewed as a fibred TY-valued 1-form ω on Y such that $\omega_v = 0$ and $\omega_h = id_{\pi^* TM}$,

* This paper is in final form and no version of it will be submitted for publication elsewhere.

see [6] . This form will be denoted by Γ_h and called the horizontal form of Γ. In coordinates $\Gamma_h = dx^i \otimes \partial/\partial x^i + \Gamma^{\alpha}_i(x,y)dx^i \otimes \partial/\partial y^{\alpha}$ where the local functions Γ^{α}_i will be called the Christoffels of Γ.

Let ω be an arbitrary fibred 1-form on Y. To find the conditions for ω to determine a connection Γ on Y let us consider the linear morphism $\omega^o : VY \otimes T^*M \to VY \otimes T^*M$ of the expression $x \mapsto \omega_v \cdot x - x \cdot \omega_h$, where the dot denotes the composition of the maps given by x, ω_v, ω_h.

Lemma 1. Every fibred TY-valued 1-form ω on Y such that ω^o is regular determines a connection on Y.

Proof. Consider the linear morphisn $b_{\omega} : x \mapsto \omega \cdot x - x \cdot \omega_h$ on $TY \otimes T^*M$. It is of the expression

$$(1) \qquad \bar{x}^i_t = a^i_j x^j_t - x^i_s a^s_t , \quad \bar{y}^{\alpha}_t = a^{\alpha}_j x^j_t + (a^{\alpha}_{\beta} x^{\beta}_t - x^{\alpha}_s a^s_t).$$

This means that if ω^o is regular then there exists a unique $x_o \in C^{\infty} TY \otimes T^*M$ such that $T\pi \cdot x_o = \mathrm{id}_{\pi^* TM}$ and $b_{\omega}(x_o) = 0$. By (1) the coordinates $(x^i_j = \delta^i_j, x^{\beta}_s)$ of x_o are $x^{\alpha}_s = -\phi^{\alpha t}_{s\beta} a^{\beta}_t$, where $\phi^{\beta t}_{s\alpha}$ are the components of the tensor field which is determined by the inverse map to ω^o. Obviously x_o is the horizontal form of the connection on Y with the Christoffels $\Gamma^{\alpha}_s = - \phi^{\alpha t}_{s\beta} \cdot a^{\beta}_t$. QED.

The connection determined by the form x_o discribed in the proof of Lemma 1 will be denoted by Γ_{ω} . Let $C^{\infty}_{\Gamma}(T^*Y \otimes TY)$ be the space of all fibred TY-valued 1-forms ω on Y such that ω^o is regular. Using the theory of natural fibre operators, see [5] , it is easy to prove that only in the case of $\omega \in C^{\infty}_{\Gamma}(T^*Y \otimes TY)$ there is a natural fibre operator D of 0-order such that $D(\omega)$ is a connection on Y and that every 0-order natural operator from $C^{\infty}_{\Gamma}(T^*Y \otimes TY)$ into the space of all connections on Y is of the form $\omega \mapsto \Gamma_{\omega}$.

Lemma 2. Let ω be a fibred TY-valued 1-form on Y. Let $A_h = \{a^1_1, \ldots, a^m_m\}$, $B_v = \{b^1_1, \ldots, b^n_n\}$ be the spectras of the linear morphisms ω_h, ω_v at $y \in Y$. Then ω^o is regular at

$y \in Y$ if and only if A_h and B_v are disjoint.

Proof. At $y \in Y$ there are bases in V_yY and in $(\pi^* TM)y$ in which the matrices of ω_h and ω_v are of the Jordan's form, i.e. $\omega^o(x) = (b^\alpha_\alpha - a^i_i)x^\alpha_i + b^\alpha_{\alpha+1}\, x^{\alpha+1}_i - a^{i-1}_i\, x^\alpha_{i-1}$. Now, it is easy to see that ω^o is regular if and only if $b^\alpha_\alpha \neq a^i_i$ for any values of α and i.

Corollaries. 1. If $\omega_h = 0$ or $\omega_v = 0$ then ω^o is regular if and only if ω_v or ω_h is regular, respectively. In these cases according to (1) $\Gamma^\alpha_i = a^\alpha_\beta\, a^\beta_i$, $a^\alpha_\beta\, \tilde{a}^\beta_\gamma = \delta^\alpha_\gamma$, or $\Gamma^\alpha_i = a^\alpha_t\, \tilde{a}^t_i$, $a^i_t\, \tilde{a}^t_j = \delta^i_j$, respectively, are the Christoffels of Γ_ω .

2. If ω^o is regular then at least one of the maps ω_h, ω_v is regular.

3. If ω_h or ω_v is regular then ω^o is regular if and only if $\lambda = 1$ is not the eingenvalue of the linear operator $u \mapsto \omega_v.u.\tilde{\omega}_h$ or $u \mapsto \tilde{\omega}_v.u.\omega_h$, respectively, on $VY \otimes T^*M$.

2. (B, V) - structures

A fibred manifold $\pi: Y \to M$ is said to be a (B,V)-structure and denoted by (Y, ε) if there is a cross-section $\varepsilon: Y \to$ $\to VY \otimes T^*M$. Throughout this paper, ε is viewed both a VY-value 1-form on Y and a linear morphism from $\pi^* TM$ into VY over id_Y.

Let us recall the Frolicher-Nijenhuis bracket of two tangent vector valued forms which is in the case of 1-forms of the form, (see [4]), $[L,K](X,Y) = [LX, KY] + [LX, KY] + LK[X,Y] + KL[X,Y] - L[KX,Y] - L[X,KY] - K[LX,Y] - K[X,LY]$.

Let $\omega = a^i_j dx^j \otimes \partial/\partial x^i + (a^\alpha_i\, dx^i + a^\alpha_\beta\, dy^\beta) \otimes \partial/\partial y^\alpha$ be a fibred TY-value 1-form on (Y, ε), $\varepsilon = \varepsilon^\alpha_i\, dx^i \otimes \partial/\partial y^\alpha$. Then $[\varepsilon, \omega]$ is called ε-torsion of ω. In coordinates we get

$$(2) \qquad [\varepsilon, \omega] = -\, a^i_{j,\beta}\, \varepsilon^\beta_s\, dx^j \wedge dx^s \otimes \partial/\partial x^i + [(\varepsilon^\alpha_t\, a^t_{j,s} -$$

$$- a^\alpha_{j,\gamma}\, \varepsilon^\gamma_s - \varepsilon^\alpha_{j,t} a^t_s - \varepsilon^\alpha_{j,\gamma}\, a^\gamma_s + a^\alpha_\beta\, \varepsilon^\beta_{j,s})dx^j \wedge dx^s +$$

$$+ (\varepsilon^{\alpha}_{s}\, a^{s}_{j,\beta} + a^{\alpha}_{\beta}{}_{,\gamma}\, \varepsilon^{\gamma}_{j} - \varepsilon^{\alpha}_{j,\gamma}\, a^{\gamma}_{\beta} + a^{\alpha}_{\gamma}\, \varepsilon^{\gamma}_{j\beta}) dx^{j} \wedge dy^{\beta}] \otimes$$

$$\otimes\, \partial/\partial y^{\alpha} ,$$

$$(3) \qquad \frac{1}{2} [\varepsilon , \varepsilon] = \varepsilon^{\alpha}_{j,\gamma}\, \varepsilon^{\gamma}_{s}\, dx^{s} \wedge dx^{j} \otimes \partial/\partial y^{\alpha} ,$$

Where we use throughout this paper the designations $\frac{\partial f_u}{\partial y^{\alpha}} := f_{u,\alpha}$, $\frac{\partial f_u}{\partial x^{j}} := f_{u,j}$. This is immediate from (2) that if ω is projectable then $[\varepsilon , \omega]$ is a VY-value 2-form and that the restriction of $[\varepsilon , \omega]$ to VY vanishes.

Remark 1. A vertical vector field Z on Y is called ε-basic if there is a vector field X on M such that $Z = \varepsilon(X)$. Let $v_1, v_2 \in T_xM$. Let X_1, X_2 be local vector fields on M such that $X_i(x) = v_i$, $i = 1, 2$. Then $\varepsilon(X_i)$ is a local ε-basic vector field on Y. Let $y \in Y_x$. Put $\varphi_y(v_1, v_2) = [\varepsilon(X_1), \varepsilon(X_2)]_y$. Calculating it and comparing with (3) we get $\varphi = \frac{1}{2}[\varepsilon , \varepsilon]$. It means that $[\varepsilon , \varepsilon] = 0$ if and only if $[Z_1, Z_2] = 0$ for any ε-basic vector fields Z_1, Z_2 on Y.

3. Vector fields and connections on (Y, ε)

Let $X = c^{i}(x,y)\,\partial/\partial x^{i} + b^{\alpha}(x,y)\,\partial/\partial y^{\alpha}$ be a vector field on Y. Being a crossection $Y \xrightarrow{X} TY$, X determines a linear morphism $X_B := pr_2 . V(T\pi . X) : VY \to TM$, $VY \to \pi^{*}TM$, $(x^{i}, y^{\alpha}, 0, dy^{\alpha}) \mapsto (x^{i}, dx^{i} = c^{i}{}_{,\beta}\, dy^{\beta})$, where $V(T\pi . X)$ denotes the vertical prolongation of the map $T\pi . X : Y \to TM$ and $pr_2 : VTM \equiv TM \times_M TM \to TM$ is the projection on the second factor. In the only case of a projectable vector field X on Y, $X_B = 0$.

The straightforward calculation of the Lie derivation of ε by X

$$(4) \qquad L_X \varepsilon = - c^{i}_{,\beta}\, \varepsilon^{\beta}_{j}\, dx^{j} \otimes \partial/\partial x^{i} + (A^{\alpha}_{i}\, dx^{i} + \varepsilon^{\alpha}_{j}\, c^{j}_{,\beta}\, dy^{\beta}) \otimes$$

$$\otimes\, \partial/\partial y^{\alpha} ,$$

$$A_i^\alpha = \mathcal{E}_j^\alpha c^j_{,i} + \mathcal{E}^\alpha_{i,j} c^j + \mathcal{E}^\alpha_{i,\beta} b^\beta - b^\alpha_{,\beta} \mathcal{E}_i^\beta$$

gives

Lemma 3. $L_X\mathcal{E}$ is a fibre TY-value 1-form on Y such that $(L_X \mathcal{E})_h = - X_B \cdot \mathcal{E}$, $(L_X\mathcal{E})_v = \mathcal{E} . X_B$.

Denote by $C^\infty_\Gamma(Y, \mathcal{E})$ the set of all vector fields X on $(Y, \mathcal{E})$ such that $L_X\mathcal{E} \in C^\infty_\Gamma(T^*Y \otimes TY)$, i.e. that $(L_X\mathcal{E})^o$ is regular. If $X \in C^\infty_\Gamma(Y, \mathcal{E})$, then Γ_X is the abbreviated notation for $\Gamma_{L_X\mathcal{E}}$. According to (1) the Christoffels of Γ_X satisfy

$$(5) \qquad (\mathcal{E}_j^\alpha c^u_{,\beta} \delta^s_t + \delta^{\alpha}_\beta c^s_{,\gamma} \mathcal{E}_t^\gamma) \Gamma^\beta_s = - A^\alpha_t .$$

If X is a projectable vector field on Y then $(L_X\mathcal{E})^o = 0$ and $X \notin C^\infty_\Gamma(Y, \mathcal{E})$. It is clear that if $X \in C^\infty_\Gamma(Y, \mathcal{E})$ then $X + Z \in C^\infty_\Gamma(Y, \mathcal{E})$ for any projectable vector field Z on Y, i.e. X is an operator $Z \mapsto \Gamma_{X+Z}$ from the space of all projectable vector fields Z into the space of all connection on Y. The expression $u \mapsto \mathcal{E}_j^\alpha c^j_{,\beta} u^\beta_s + u^\alpha_j c^j_{,\gamma} \mathcal{E}^\gamma_s$ of $(L_X\mathcal{E})^o$ induces some special cases. If $\dim M < \dim Y_x$ then $\mathcal{E} . X_B$ is not regular, i.e. $X \in C^\infty_\Gamma(Y, \mathcal{E})$ implies that $X_B \cdot \mathcal{E}$ is regular. Certainly, if $X_B . \mathcal{E} = id_{\pi^* TM}$ then $X \in C^\infty_\Gamma(Y, \mathcal{E})$ if and only if the operator $u \mapsto \mathcal{E} . X_B . u$ on $VY \otimes T^*M$ has not the eingenvalue $- 1$. Quite analogously if $\dim M > \dim Y_x$ then $X_B . \mathcal{E}$ is not regular and the regularity of $\mathcal{E} . X_B$ is a necessary condition for X to belong to $C^\infty_\Gamma(Y, \mathcal{E})$.

Example 1. There is the canonical (B,V)-structure on TM given by the canonical morphism $\mathcal{E} = dx^i \otimes \partial/\partial x^i_1$ on TM with a chart (x^i, x^i_1). In this case $(L_X\mathcal{E})_h = - X_B = -(L_X\mathcal{E})_v$. Then, by Lemma 2, $X \in C^\infty_\Gamma(TM, \mathcal{E})$ if and only if X_B is regular.

Let $\dim M = \dim Y_x$. Let $\mathcal{E} : \pi^* TM \to VY$ be an isomorphism. A vector field X on $(Y, \mathcal{E})$ is said to be conjugated with $\mathcal{E}$ if $X_B = \mathcal{E}^{-1}$. There is an isomorphism $\mathcal{E}$ that does not admit a vector field conjugated with $\mathcal{E}$. To show it we constructe an object $\mathcal{E}^{-1}_v$. Let $\mathcal{E} = \mathcal{E}^\alpha_i dx^i \otimes \partial/\partial y^\alpha$ be an isomorphism. Then $\mathcal{E}^{-1} : VY \to TM$ is a morphism over π. Its expression in charts $(x^i, y^\alpha, 0, \gamma^\alpha)$ on VY and (x^i, x^i_1) on TM is $\bar{x}^i = x^i$,

$x_1^i = \tilde{\mathcal{E}}^i \tau^\alpha$, where $\mathcal{E}_i^\alpha \tilde{\mathcal{E}}_\beta^i = \delta_\beta^\alpha$. Let $V\mathcal{E}^{-1}$ be the vertical differential of $\mathcal{E}^{-1}$ according to the submersion $VY \to M$, $V\mathcal{E}^{-1}(x^i, y^\alpha, 0, \tau^\alpha, dx^i = 0, dy^\alpha, 0, d\tau^\alpha) = (x^i, dx_1^i = \tilde{\mathcal{E}}^i_{\alpha,\beta} \tau^\alpha dy^\beta + \tilde{\mathcal{E}}^i_\alpha d\tau^\alpha)$. Recall the canonical involution $i_2(x^i, y^\alpha, \xi^i, \tau^\alpha, dx^i, dy^\alpha, d\xi^i, d\tau^\alpha) = (x^i, y^\alpha, dx^i, dy^\alpha, \xi^i, \tau^\alpha, d\xi^i, d\tau^\alpha)$ on TTY. Then $V\mathcal{E}^{-1}.i_2(x^i, y^\alpha, 0, \tau^\alpha, dx^i = 0, dy^\alpha, 0, d\tau^\alpha) = (x^i, dx_1^i = \tilde{\mathcal{E}}^i_{\alpha,\beta} dy^\alpha \tau^\beta + \mathcal{E}^i_\alpha d\tau^\alpha)$. Let $\tau_1, \tau_2 \in VY$. There is $\tau \in VVY$ such that $p_{TY}(\tau) = \tau_1$, $p_{TY}(i_2\tau) = \tau_2$, where $p_{TY}: TTY \to TY$ is the tangent projection. Put $\mathcal{E}_v^{-1}(\tau_1, \tau_2) := (V\mathcal{E}^{-1} - V\mathcal{E}^{-1} i_2)(\tau) = (x^i, \tilde{\mathcal{E}}^i_{\alpha,\beta}(\tau_1^\alpha \tau_2^\beta - \tau_2^\alpha \tau_1^\beta))$, i.e. $\mathcal{E}_v^{-1} \in C^\infty(\wedge^2 VY \otimes TM)$.

Lemma 4. Let X be vector field conjugated with $\mathcal{E}$. Then $\mathcal{E}_v^{-1} = 0$.

Proof. Let $X = c^i \partial/\partial x^i + b^\alpha \partial/\partial y^\alpha$. Then $c^i_{,\alpha} = \tilde{\mathcal{E}}^i_\alpha$ and thus $\tilde{\mathcal{E}}^i_{\alpha,\beta} = \tilde{\mathcal{E}}^i_{\beta,\alpha}$. It completes our proof.

Lemma 5. If a vector field is conjugated with $\mathcal{E}$ then $X \in C^\infty_r(Y, \mathcal{E})$.

Proof. In this case $(L_X\mathcal{E})^o = 2id_{VY \otimes T^*M}$ is regular. QED.

If X is conjugated with $\mathcal{E}$ then by (5) the Christoffels of the connection Γ_X are of the simple form $\Gamma_i^\alpha = -\frac{1}{2} A_i^\alpha$, in virtue of (4) $(id_{TY} - L_X\mathcal{E})/2$ is the horizontal form of Γ_X and $\mathcal{E}$-torsion of Γ_X, (i.e. $[\mathcal{E}, L_X\mathcal{E}]$ is a VY-value 1-form of the expression

$$(6) \quad [\mathcal{E}, L_X\mathcal{E}] = (-A^\alpha_{j,\gamma} \mathcal{E}^\gamma_s + 2\mathcal{E}^\alpha_{j,s} - \mathcal{E}^\alpha_{j,\gamma} A^\gamma_s)\, dx^j \wedge dx^s \otimes \otimes \partial/\partial y^\alpha .$$

Proposition 1. If X is a vector field on Y such that $X_B: VE \to \pi^* TM$ is an isomorphism then X determines a connection on Y.

Proof. Denote $\mathcal{E} = X_B^{-1}$. It is clear that X is conjugated with the (B,V)-structure $(Y, \mathcal{E})$. Then Lemma 5 completes our proof.

If $X = c^i \partial/\partial x^i + b^\alpha \partial/\partial y^\alpha$ is such that X_B is an isomorphism then $\Gamma^\alpha_j = -\frac{1}{2} A^\alpha_i = \frac{1}{2}(\tilde{c}^\alpha_j c^j_{,i} + \tilde{c}^\alpha_{i,j} c^j + \tilde{c}^\alpha_{i,\beta} b^\beta - b^\alpha_{,\beta} \tilde{c}^\beta_i)$ are the Christoffels of Γ_X on $(Y, \mathcal{E} = X_B^{-1})$, where $\tilde{c}^\alpha_j c^k_{,\alpha} = \delta^k_j$. This means that the map $X \mapsto \Gamma_X$ is an operator of second order from the space of all vector fields X on Y such that X_B is an isomorphism into the space of all connections on Y.

4. Special (B,V)-structure on vector bundles.

Let $\pi: E \to M$ be a vector bundle. The canonical identification $VE \equiv E \times_M E$ states by every E-valued 1-form $\tilde{\mathcal{E}}$ on M a (B,V) structure $(E, \mathcal{E})$, (called projectable), where $\mathcal{E}(y,v) = (y, \tilde{\mathcal{E}}(v))$, $y \in E$, $v \in T_{\pi(y)}M$. In coordinates, $\mathcal{E} = \mathcal{E}^\alpha_i(x) dx^i \otimes \partial/\partial y^\alpha$. In this case according to (3) $[\mathcal{E}, \mathcal{E}] = 0$.

Let $(E, \mathcal{E})$ be a projectable (B,V)-structure such that $\mathcal{E}: \pi^* TM \to VE$ is an isomorphism. A vector field $X = c^i(x,y) . \partial/\partial x^i + b^\alpha(x,y) \partial/\partial y^\alpha$ on E is conjugated with $\mathcal{E}$ if and only if $c^i_{,\beta} = \tilde{\mathcal{E}}^i_\beta(x)$, $\tilde{\mathcal{E}}^i_\beta \mathcal{E}^\beta_j = \delta^i_j$, i.e. if and only if $c^i = \tilde{\mathcal{E}}^i_\beta(x) y^\beta + \gamma^i(x)$. Let $L = y^\alpha \partial/\partial y^\alpha$ be the Liouville vector field on E. We immediately get

Proposition 2. Let X be a vector field on a projectable (B,V)-structure $(E, \mathcal{E})$. Then $[V, X]$ is conjugated with $\mathcal{E}$ and every vector field on E conjugated with $\mathcal{E}$ is of the form $X + Z$, where $\mathcal{E}(X) = V$ and Z is a projectable vector field on E.

Proposition 3. Let X be a vector field on a projectable (B,V)-structure $(E, \mathcal{E})$ conjugated with $\mathcal{E}$. Then the connection Γ_X is wilhout $\mathcal{E}$-torsion, i.e. $[\mathcal{E}, L_X \mathcal{E}] = 0$.

Proof. Since $\mathcal{E}^\alpha_j c^u_{,\gamma} = \delta^\alpha_\beta$ therefore $\mathcal{E}^\alpha_{j,s} c^j_{,\gamma} = -\mathcal{E}^\alpha_j c^j_{,\gamma s} = -\mathcal{E}^\alpha_j c^j_{,s\gamma}$. Then by (4) $A^\alpha_{i\gamma} = -\mathcal{E}^\alpha_{j,i} c^j_\gamma + \mathcal{E}^\alpha_{i,j} c^j_{,\gamma} - b^\alpha_{,\beta\gamma} \mathcal{E}^\beta_i$. With respect to (6) we have $[\mathcal{E}, L_X \mathcal{E}] = (\mathcal{E}_{s,i} - \mathcal{E}_{i,s} + b^\alpha_{,\beta\gamma} \mathcal{E}^\beta_i \mathcal{E}^\gamma_s + 2\mathcal{E}^\alpha_{i,s}) dx^i \wedge dx^s \otimes \partial/\partial y^\alpha = 0$.

Remark 2. Let X be a vector field on E such that $X_B: VE \to \pi^* TM$ is an isomorphism and $(E, \mathcal{E} = X_B^{-1})$ is projectable. Then X is conjugated with $\mathcal{E}$ and by virtue of Proposition 3

Γ_X is without $\mathcal{E}$-torsion.

Example 2. Let us return to example 1. The canonical (B,V)-structure (TM, $\mathcal{E} = dx^i \otimes \partial/\partial x_1^i$) is projectable and it is induced by $\bar{\mathcal{E}} = id_{TM}$. If V is the Liouville field on TM then a vector field X on TM such that $\mathcal{E}(X) = V$ is a diffential equation of second order on M. Therefore we can reformulate Proposition 2 in the following way.

Proposition 4. A vector field X on TM is conjugated with the canonical morphism $\mathcal{E} = dx^i \otimes \partial/\partial x_1^i$ if and only if it is of the form U + Z, where U is a differential equation of second order on M and Z is a projectable vector field on TM.

In coordinates, X is concugated with $dx^i \otimes \partial/\partial x_1^i$ if and only if $X = (x_1^i + Z^i(x))\partial/\partial x^i + b^i(x,x_1)\partial/\partial x_1^i$. Then the Christoffels of Γ_X are $\Gamma^i_j = -\frac{1}{2} A^i_j$

(7) $$\Gamma^i_j = -\frac{1}{2} A^i_j = -\frac{1}{2}(\partial z^i/\partial x^j - \partial b^i/\partial x_1^j) .$$

It coincides with [1] , [2] for X being a differential equation of second order on M.

Let $Z = a^i(x)\partial/\partial x^i$ be a vector field on M. Then $TZ = a^i \partial/\partial x^i + \frac{\partial a^i}{\partial x^j} x_1^j \partial/\partial x_1^i$ is the T-prolongation of Z on TM. It is a projectable vector field on TM.

Proposition 5. Let X be a differential equation of second order on M. Let Z be a vector field on M. Then $\Gamma_{X+TZ} = \Gamma_X$.

Proof. Let $X = x_1^i \partial/\partial x^i + b^i(x,x_1)\partial/\partial x_1^i$, $Z = a^i\partial/\partial x^i$. Then by (7) $\Gamma^i_j = \frac{1}{2}\frac{\partial b^i}{\partial x_1^j}$ are the Christoffels of both Γ_{X+TZ} and Γ_X.

Another special (B,V)-structures on TM can be constructed as follows. Let X be a vector field on $p_M: TM \rightarrow M$ such that $X_B: VTM \rightarrow p_M^* TM$ is an isomorphism. Since $VTM \equiv TM \times_M TM \equiv p_M^* TM$ there are two (B,V)-structures on TM both (TM, X_B^{-1}) and (TM, X_B). We say that X is 2-homothetic if $X_B^2 = t.id_{VTM}$, $t \in R$. Every vector field $X = tW + Z$ where $t \in R$, W is a differential equation of second order and Z is a projectable vector field on TM is 2-homothetic. In coordinates, $X = c^i(x,x_1)\,\partial/\partial x^i + b^i(x,x_1)\,\partial/\partial x_1^i$ is 2-homothetic iff $(\partial c^i/\partial x_1^s)(\partial c^s/\partial x^k) = t\delta^i_k$. Then using (3) or (2) we get, respectively:

Lemma 6. If X is 2-homothetic then $[X_B, X_B] = 0$.

Proposition 6. Let X be a 2-homothetic vector field on TM. Let W be a vector field on TM conjugated with X_B. Then the connection Γ_W is without X_B-torsion.

Example 3. $\pi : T^*M \to M$.

Let (x^i, z_i) be a chart in T^*M. Then $V = z_i\,\partial/\partial z_i$, $\lambda = z_i dx^i$, $d\lambda = dz^i \wedge dx^i$ are the Liouville field, the Liouville form, the canonical symplectic form on T^*M.

Let $(T^*M,\ \mathcal{E} = \mathcal{E}_{ij}(x,z)dx^i \otimes \partial z_j)$ be a (B,V)-structure on T^*M. If $\mathcal{E} : \pi^* TM \to VT^*M$ is an isomorphism and $X = c^i(x,z)\,\partial/\partial x^i + b_i(x,z)\,\partial/\partial z_i$ is a vector field on T^*M conjugated with $\mathcal{E}$ then X determines both the connection Γ_u the Christoffels of which are $\Gamma_{ij} = -\frac{1}{2}(\mathcal{E}_{is}c^s_j + \mathcal{E}_{ij,s}c^s + \mathcal{E}^{\bar{s}}_{ij}b_s - b^{\bar{s}}_i\,\mathcal{E}_{sj}$, where $f^{\bar{s}} := \frac{\partial f}{\partial z_s}$, and the connection $d\lambda$-orthogonal to Γ_u the Christoffels of that are $\bar{\Gamma}_{ij} = \Gamma_{ji}$.

We say that $\mathcal{E}$ is symmetric if for any $X,Y \in TT^*M$ $d\lambda(\mathcal{E}X,Y) = d\lambda(\mathcal{E}Y,X)$, $\mathcal{E}_{ij} = \mathcal{E}_{ji}$.

If $\mathcal{E}$ is an isomorphism then we can constructe a function on T^*M as follows. Let X be an arbitrary vector field on T^*M such that $\mathcal{E}(X) = V$. Put $H_{\mathcal{E}} := d\lambda(V,X)$. In coordinates $H_{\mathcal{E}} = \tilde{\mathcal{E}}^{ij}z_i z_j$, $\tilde{\mathcal{E}}^{is}\mathcal{E}_{sj} = \delta^i_j$.

Let $(T^*M, \mathcal{E})$ be projectable and regular, i.e. $\mathcal{E}$ is given by an isomorphism $\bar{\mathcal{E}} : TM \to T^*M$, $\bar{\mathcal{E}} = \mathcal{E}_{ij}(x)dx^i \otimes dx^j$. By virtue of Proposition 2 every vector field on T^*M conjugated with $\mathcal{E}$ is of the form $W = (\tilde{\mathcal{E}}^{ik}z_k + \gamma^i(x))\,\partial/\partial x^i + b_i\,\partial/\partial z_i$, i.e. $W = T\bar{\mathcal{E}}(X)$ where X is a vector field on TM conjugated with the canonical (B,V)-structure $(TM,\ dx^i \otimes \partial/\partial x_1^i)$.

It is easy to verify that the vector field X on T^*M satistying the equation $i_X d\lambda = \mathcal{H}\, dH_{\mathcal{E}}$, where $\mathcal{H} \in R$ and i_X denotes the usual insertion operator, is conjugated with $\mathcal{E}$ if and only if $\mathcal{H} = -\frac{1}{2}$ and $\mathcal{E}$ is symmetric. Then the connection Γ_X is the just connection induced on T^*M by the Levi-Civita connection on TM determined by the regular symmetric bilinear form $\bar{\varepsilon}$ on M.

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