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GENERAL POISSON ALGEBRAS

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1. Introduction.

In contact and symplectic geometries the algebra $C^{\infty}(M)$ of smooth functions on a given manifold M is additionally furnished with a Lie algebra structure given by the corresponding Poisson bracket.
These Lie algebras are closely related to the Lie algebras of contact and hamiltonian vector fields on M. More precisely, let (M,ω) be a C^{∞} symplectic manifold, i.e. ω is a closed non-degenerated 2-form on M.
The symplectic form ω induces the isomorphism $\mu : TM \longrightarrow T^{*}M$ of the tangent and cotangent bundles defined by $\mu(X) = -i_X\omega$, where $i_X\omega$ denotes the interior multiplication.
Put $\hat{f} = \mu^{-1}(df)$ for $f \in C^{\infty}(M)$. The vector fields of the form $\hat{f}$ are called hamiltonian and they can be also described as those vector fields X for which $i_X\omega$ is an exact 1-form.
The family $\Gamma_{\omega}(M)$ of all hamiltonian vector fields is a Lie subalgebra in the Lie algebra of all smooth vector fields.
The Poisson bracket $(\, , \,)$ in $C^{\infty}(M)$ defined by $(f,g) = \hat{f}(g)$ is a Lie bracket, i.e. it makes $C^{\infty}(M)$ into a Lie algebra. Moreover, the mapping $\hat{}: C^{\infty}(M) \longrightarrow \Gamma_{\omega}(M)$ is a Lie algebra surjective homomorphism with the kernel consisting of constant (locally) functions.
The Poisson bracket can be also expressed in the form $(f,g) = \Omega(df \wedge dg)$, where Ω is a 2-vector field corresponding to the 2-form ω via the isomorphism of tensor bundles induced by μ.
Since vector fields act on $C^{\infty}(M)$ as derivations of the associative algebra, it is easy to prove the formula

(1) $\quad (f,gh) = (f,g)h + g(f,h)$.

Such objects were investigated from an algebraic point of view in [1] and [4] .
Let now (M,β) be a contact C^{∞}-manifold of dimension $2n + 1$, i.e. β is a 1-form such that $\beta \wedge (d\beta)^n$ does not vanish on M. Then we have the splitting $TM = Ker(\beta) \oplus Ker(d\beta)$, where
$Ker(\beta) = \{ X \in TM : i_X\beta = 0 \}$ and $Ker(d\beta) = \{ X \in TM : i_X d\beta = 0 \}$

are vector bundles of dimensions 2n and 1. Notice that $Ker(d\beta)$ is generated by the unique vector field Y satisfying $i_Y\beta = 1$ and $i_Y d\beta = 0$. The vector bundle homomorphism

$$\mu : TM \ni X \longmapsto i_X d\beta \in T^*M$$

restricted to $Ker(\beta)$ is an isomorphism onto a subbundle B of T^*M. We have the splitting $T^*M = B \oplus \mathbb{R}\beta$ and the projection $P: T^*M \longrightarrow B$. The mapping $f \mapsto \hat{f} = \mu^{-1}(P(df)) + fY$ is a linear isomorphism of the space $C^\infty(M)$ onto the Lie algebra $\Gamma_\beta(M)$ of contact vector fields, i.e. the vector fields satisfying $L_X\beta = f_X\beta$ for some $f_X \in C^\infty(M)$ (L_X is the Lie derivative along X).
The contact Poisson bracket in $C^\infty(M)$ is then defined by $(f,g) = \hat{f}(g) - gY(f)$ and it makes $C^\infty(M)$ into a Lie algebra such that the mapping $\hat{}: C^\infty(M) \longrightarrow \Gamma_\beta(M)$ is a Lie algebra isomorphism.
One can check that we have a similar to (1) formula

(2) $(f,gh) = g(f,h) + (f,g)h + gh(1,f)$

which becomes identically (1) if $ad_1 = 0$.
Since $d\beta$ is non-degenerated on $Ker(\beta)$, it corresponds via the homomorphism μ to a 2-vector field Ω such that $i_\beta\Omega = 0$ and the contact Poisson bracket can be also written in the form

(3) $(f,g) = \Omega(df \wedge dg) + fY(g) - gY(f)$.

All this was generalized by A.A.Kirillov [6] who showed that every local Lie algebra of one-dimensional bundle (i.e. the space $\Gamma(E)$ of smooth sections of a one-dimensional vector bundle E over a manifold M, equipped with a Lie bracket (,) such that supp((f,g)) is contained in $supp(f) \cap supp(g)$) is of this type. More precisely, since $\Gamma(E)$ is locally the space of smooth functions, there are a 2-contravariant tensor field Ω and a vector field Y such that locally the Lie bracket in $\Gamma(E)$ is of the form (3).
It is clear that such Ω and Y should satisfy some additional conditions to define a Lie bracket (see §.3.).
Since the formula (2) is purely algebraic, it allows us to propose the following definition.

Definition. A general Poisson algebra is an associative commutative algebra A with unit element **1** furnished additionally with a Lie bracket (,) such that (2) holds true for all $f,g,h \in A$.

Remark that we were recently informed that such structures had been also investigated by S.M.Skriabin in his thesis [11].
It is interesting that the formula (2) holds true also in some non-commutative cases, namely for associative (non-commutative) algebras with the natural Lie bracket $(X,Y) = XY - YX$ (see [4]).

In §.3. we will show that the formulas (2) and (3) are in fact equivalent even in a purely algebraic sense .

2. Other examples.

Another example of a general Poisson algebra is the algebra $C^{\infty}(M)$ with the bracket $(f,g)_Y = fY(g) - gY(f)$ for Y being a vector field on M . It is a particular case of (3) with $\Omega = 0$, but it is rather important, since it appears when you work with Lie algebras of vector fields which are $C^{\infty}(M)$-modules, e.g. the Lie algebras of vector fields tangent to a given generalized foliation (see [3] and [5]). Indeed, for a vector field Y and $f,g \in C^{\infty}(M)$ we have $[fY,gY] = (f,g)_Y Y$.

Finally, let us consider an example from unimodular geometry. Let (M,η) be an unimodular manifold of dimension n , i.e. η is a nowhere-vanishing n-form on M . We have the bundle isomorphism $\mu : TM \ni X \longmapsto i_X\eta \in \Lambda^{n-1}T^*M$. The Lie algebra $\hat{\Gamma}_\eta(M)$ of divergence free vector fields corresponds via this isomorphism to the space of closed $(n-1)$-forms. The derived ideal $\Gamma_\eta(M) = [\hat{\Gamma}_\eta(M), \hat{\Gamma}_\eta(M)]$ consists of vector fields corresponding to exact $(n-1)$-forms. Thus we have the surjective mapping $\sim : \Omega^{n-2}(M) \longrightarrow \Gamma_\eta(M)$ from the space $\Omega^{n-2}(M)$ of $(n-2)$-forms onto $\Gamma_\eta(M)$ defined by $\tilde{\alpha} = \mu^{-1}(d\alpha)$ with the kernel consisting of closed $(n-2)$-forms.

The $C^{\infty}(M)$-module $\Omega^{n-2}(M)$ is finitely generated by the forms $\alpha = df_1 \wedge \ldots \wedge df_{n-2}$, where $f_1, \ldots, f_{n-2} \in C^{\infty}(M)$. For such an α the bracket $(\, , \,)_\alpha$ defined on $C^{\infty}(M)$ by $(f,g)_\alpha = (f\alpha)^{\sim}(g)$ is a Lie bracket, and since $[(f\alpha)^{\sim}, (g\alpha)^{\sim}] = ((f,g)_\alpha \alpha)^{\sim}$, the mapping

$$\hat{} : C^{\infty}(M) \ni f \longrightarrow \hat{f} = (f\alpha)^{\sim} \in \Gamma_\eta(M)$$

is a Lie algebra homomorphism.

It is easy to see that $C^{\infty}(M)$ with $(\, ,)_\alpha$ is a general Poisson algebra for which $ad_1 = 0$ and hence $\Gamma_\eta(M)$ is a finite sum (non-direct in general) of quotients of general Poisson algebras.

3. The Kirillov formula for general Poisson algebras.

Suppose that A with the Lie bracket $(\, , \,)$ is a general Poisson algebra. Observe that ad_1 is a derivation of the associative algebra A , since by (2) $(1,gh) = (1,g)h + (1,h)g + gh(1,1)$ and the last term in the sum equals zero. Denote $\alpha = ad_1$. It is easy to verify that for each $f \in A$ the mapping $D_f : A \ni g \longmapsto (f,g) + \alpha(f)g \in A$ is also a derivation of A . Hence the skew-symmetric bilinear $\omega : A \times A \longrightarrow A$ given by $\omega(f,g) = D_f(g) - f\alpha(g)$. satisfies $(f,g) = \omega(f,g) + f\alpha(g) - g\alpha(f)$ and $\omega(f,\cdot)$ is a derivation of

A for each $f \in A$. This is exactly the Kirillov form of the Lie bracket for local Lie algebras of one-dimensional bundles.
To see what are the relations between ω and α, it is convenient to introduce some notions. The ideas are in fact well-known and they can be found in [2] (cf. also [7]).

Let A be a commutative associative algebra with unit element $\mathbf{1}$ over a field K of characteristic zero, and let $V_p(A)$, $p \geq 1$, be the space of all p-linear antisymmetric mappings
$\alpha : A \times \ldots \times A \longrightarrow A$.
Denote $V_0(A) = A$, $V_p(A) = \{0\}$ for $p < 0$, and $V(A) = \bigoplus_{p=-\infty}^{p=+\infty} V_p(A)$.
V(A) with the exterior multiplication

$$\alpha \wedge \beta (f_1, \ldots, f_{a+b}) = \\ = \frac{1}{a!b!} \sum_{s \in S(a+b)} \mathrm{sgn}(s) \alpha(f_{s(1)}, \ldots f_{s(a)}) \beta(f_{s(a+1)}, \ldots, f_{s(a+b)}),$$

where $\alpha \in V_a(A)$, $\beta \in V_b(A)$, and $S(a+b)$ stands for the symmetric group, is a graded commutative algebra, i.e. the multiplication is associative and we have $\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha$.
The exterior derivative $d : V_p(A) \longrightarrow V_{p+1}(A)$, defined by

$$d\alpha(f_1, \ldots, f_{p+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} f_i \, \alpha(f_1, \ldots, \hat{f}_i, \ldots, f_{p+1})$$

satisfies $d^2 = 0$ and it is a graded derivation of V(A) of degree one, since $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a \alpha \wedge d\beta$ for $\alpha \in V_a(A)$.
Given $\alpha \in V_a(A)$, $\beta \in V_b(A)$, the interior multiplication

$$i_\alpha \beta(f_1, \ldots, f_{a+b-1}) = \\ = \frac{1}{a!(b-1)!} \sum_{s \in S} \mathrm{sgn}(s) \beta(\alpha(f_{s(1)}, \ldots, f_{s(a)}), f_{s(a+1)}, \ldots, f_{s(a+b-1)})$$

defines a graded derivation $i_\alpha : V(A) \longrightarrow V(A)$ of degree $(a-1)$,
The graded commutator $[i_\alpha, i_\beta] = i_\alpha \circ i_\beta - (-1)^{(a-1)(b-1)} i_\beta \circ i_\alpha$ equals $i_{[\alpha,\beta]}$, where $[\alpha, \beta] = i_\alpha \beta - (-1)^{(a-1)(b-1)} i_\beta \alpha$.
One can prove that V(A) equipped with the bracket $[\ ,\ ]$ becomes a graded Lie algebra, i.e. $[\ ,\ ]$ is bilinear, graded anticommutative ($[\alpha, \beta] = -(-1)^{(a-1)(b-1)} [\beta, \alpha]$), and satisfies the graded Jacobi identity $[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{(a-1)(b-1)} [\beta, [\alpha, \gamma]]$.
Consider in the space $V_p(A)$ the subspace $\mathrm{Der}_p(A)$ consisting of p-linear derivations of A, i.e. $\alpha \in \mathrm{Der}_p(A)$ if and only if $i_{fg}\alpha = (fi_g + gi_f)\alpha$ for all $f, g \in A$. In particular, $\mathrm{Der}_1(A)$ is the space of derivations of A.
It is not hard to verify that the space $\mathrm{Der}_*(A) = \bigoplus_{p=-\infty}^{p=+\infty} \mathrm{Der}_p(A)$ is a graded commutative subalgebra and a graded Lie subalgebra of

$V(A)$ .It can be called the Schouten-Nijenhuis algebra of A ,since in the case $A = C^{\infty}(M)$ one can identify $(Der_*(A), [\,,\,])$ with the space of skew-multivector fields with the Schouten-Nijenhuis bracket (cf. [9], [10], [12]) .For $\alpha \in Der_1(A)$ the mapping $L_\alpha = ad_\alpha$ is a graded derivation of $V(A)$ of degree 0 called the Lie derivative along α .Note also that for $\alpha \in Der_*(A)$ we have

(4) $\quad i_1\alpha = 0$ and $i_1 d\alpha = \alpha$.

The Jacobi identity has a simple expression in terms of the Schouten -Nijenhuis bracket,namely for $\omega \in V_2(A)$ we have (cf. [8])

$$2(\omega(f,\omega(g,h)) + \omega(g,\omega(h,f)) + \omega(h,\omega(f,g))) = [\omega,\omega](f,g,h) .$$

Thus a skew-symmetric bilinear mapping $\omega : A \times A \longrightarrow A$ defines a Lie bracket in A if and only if $[\omega,\omega] = 0$.

We showed that for each general Poisson algebra A there are $\omega \in Der_2(A)$ and $\alpha \in Der(A)$ such that

$$(f,g) = \omega(f,g) + f\alpha(g) - g\alpha(f) .$$

In the introduced language $f\alpha(g) - g\alpha(f) = d\alpha(f,g)$,so $(\,,\,) = \omega + d\alpha$ and from the Jacobi identity we get $[\omega+d\alpha,\omega+d\alpha] = 0$. One can prove that for $\omega \in Der_2(A)$ and $\alpha \in Der(A)$ we have $[d\alpha,d\alpha] = 0$ and $[\omega,d\alpha] = 3\omega\wedge\alpha + d[\alpha,\omega]$,that implies

(5) $\quad 0 = [\omega+d\alpha,\omega+d\alpha] = [\omega,\omega] + 6\omega\wedge\alpha + 2d[\alpha,\omega]$.

But $[\alpha,\omega]$, $\omega\wedge\alpha$, $[\omega,\omega] \in Der_*(A)$ and by (4) and (5)

$$0 = i_1([\omega,\omega] + 6\omega\wedge\alpha) + 2i_1 d[\alpha,\omega] = 0 + 2[\alpha,\omega] .$$

Hence (5) is equivalent to the system of equalities

$$[\alpha,\omega] = L_\alpha\omega = 0 \quad \text{and} \quad [\omega,\omega] + 6\omega\wedge\alpha = 0 ,$$

that proves the following.

<u>Theorem</u>. Let A be an associative commutative algebra with unit element over a field of characteristic zero.
Then A with the bracket $(\,,\,)$ is a general Poisson algebra if and only if $(\,,\,) = \omega + d\alpha$ for some $\omega \in Der_2(A)$, $\alpha \in Der(A)$ satisfying the equalities

i) $\quad L_\alpha\omega = 0$

and

ii) $\quad [\omega,\omega] + 6\omega\wedge\alpha = 0$.

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