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INTEGRAL GEOMETRY FOR COMPLEXES OF RATIONAL CURVES

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The aim of this lecture is to obtain a local inversion formula for the problem of integral geometry for complexes of rational curves (i.e. n-parameter families of rational curves on an n-dimensional complex manifold). Actually, it is the most general type of complexes of curves for which a local inversion formula exists. However, in the sequel we make no discussion of the necessity of that condition referring the reader to [1,3]. Instead, we concentrate on finding an explicit form of the inversion formula and consider how the geometry of a family influences its structure.

1. Radon transform in $\mathbb{C}^2$. Let $\varphi(z_1, z_2)$ be a finite C^∞-function on $\mathbb{C}^2$. Consider its integrals over complex lines $z_2 = \xi_1 z_1 + \xi_2$ (the complex Radon transform):

$$(1)\quad \hat{\varphi}(\xi_1, \xi_2) = \int_{\mathbb{C}^1} \varphi(z_1, \xi_1 z_1 + \xi_2)\, dz_1 \wedge d\bar{z}_1.$$

Then there is the following inversion formula (see, e.g. [2,4]):

$$(2)\quad \hat{\varphi}(w_1, w_2) = c \int_{\mathbb{C}^1} \frac{\partial^2}{\partial \xi_2 \partial \bar{\xi}_2} \varphi(\xi_1, \xi_2) \Big|_{\xi_2 = w_2 - \xi_1 w_1} d\xi_1 \wedge d\bar{\xi}_1,$$

where $c = 1/2\pi^2$.

Thus, for each line going through w one first applies a differential operator acting along a family of parallel lines, and then takes the mean value for the family of lines going through w.

There is an important difference between the Radon transform for complex lines and the Radon transform for real lines. Namely, for the complex case the inversion formula is a local one, i.e. in order to recover a value of a function at a given point it is sufficient to know only those integrals that are computed along the

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This paper is in final form and will not be submitted for publication elsewhere lines close to that point (i.e. infinitesimally).

2. Formulation of the 2-dimensional problem. We are interested in those families of curves (2-parameter families on a 2-dimensional manifold), for which a function can be reconstructed from its integrals by a local formula similar to (2). All our considerations will be local and in general position. Thus one can assume that we have a sufficiently small neighbourhood $D \subset \mathbb{C}^2$ and a 2-parameter holomorphic family of holomorphic curves E_ξ, $\xi \in \Xi$, in D. One can assume that E is defined by an equation in D. One can assume that E_ξ is defined by an equation in $\mathbb{C}^4(z,\xi)$:

$$(3) \qquad \Phi(z_1,z_2;\xi_1,\xi_2) = 0.$$

We also assume that the whole parameter domain Ξ is lying in $\mathbb{C}^2_\xi$. Since we consider a local situation, one can take z_1 as a parameter for all curves in E_ξ, i.e. E_ξ is of the form

$$(4) \qquad z_2 = f(z_1;\xi).$$

Let a family of densities $\Psi(z_1;\xi)$ be given on E_ξ. Define an integral transform for $\varphi \in C_0^\infty(D)$ by the formula

$$(5) \quad \hat{\varphi}(\xi) = \int_{E_\xi} \varphi(z_1,z_2)\,\Psi(z_1;\xi)\,\overline{\Psi(z_1;\xi)}\, dz_1 \wedge d\bar{z}_1 .$$

As we have already mentioned above, we are interested in the question when the integral transform $\varphi \mapsto \hat{\varphi}$ admits a local inversion formula? More precisely, when there exists such a (holomorphic) differential operator

$$(6) \quad L_w = \ell_1(w,\xi)\frac{\partial}{\partial\xi_1} + \ell_2(w,\xi)\frac{\partial}{\partial\xi_2} + \ell_0(w,\xi),$$

$$L_w^0 = \ell_1\frac{\partial}{\partial\xi_1} + \ell_2\frac{\partial}{\partial\xi_2},$$

that

$$(7) \quad \varphi(w) = c\int_{\gamma_w} L_w\bar{L}_w\,\hat{\varphi}(\xi)\; d\xi_1 \wedge d\bar{\xi}_1,\quad c = 1/2\pi^2,$$

where γ_w is the family of curves E_ξ going through $w \in D$ under the assumption that γ_w is a cycle. Note that those $\xi \in \Xi$ for which

$w \in E_\xi$ form a curve (evidently, in general position) γ_w on Ξ. Thus there arises a dual 2-parameter family of curves γ_w parameterized by the points $w \in D$.

A triple (Ξ, Ψ, L) where Ξ defines the family of curves E_ξ, Ψ the set of densities on them, and L differential operators L_w is called admissible if condition (7) is satisfied.

3. Admissibility conditions. Let

$$(8)\quad M_w = \frac{\partial}{\partial \xi_1} + m(w,\xi)\frac{\partial}{\partial \xi_2}$$

be a derivation along the tangent vector to the curve γ_w at the point ξ. Let

$$(9)\quad p(z_1;w,\xi) = M_w f,\ q(z_1;w,\xi) = L_w^0 f,$$

$$\mu(z_1;w,\xi) = \frac{q(z_1;w,\xi)}{p(z_1;w,\xi)} .$$

Note that for $z_1 = w_1$ p has zero and μ has a pole.

Theorem. A triple (Ξ,Ψ, L) is admissible if

$$\text{(i)}\quad M_w(\mu(z_1;w,\xi)\,\Psi(z_1,\xi)) = L_w\Psi(z_1,\xi).$$

$$\text{(ii)}\quad (z_1 - w_1)\mu\Psi \to 1 \qquad \text{when} \quad z_1 \to w_1$$

for all z_1, w, ξ.

Note that for the family of lines considered in section 1 one has $\Psi \equiv 1$, $L_w = \frac{\partial}{\partial \xi_2}$, $\mu(z_1; w,\xi) = 1/(z_1-w_1)$. Therefore, both sides of (i) vanish and hence the conditions are satisfied.

Actually, the conditions of the theorem are not only sufficient but also necessary ones. On their necessity see [3]. We only note that one can formally obtain (i) by substituting the delta function $\delta(z-\tilde{w})$, $\tilde{w} \neq w$ into (7) and by requiring that the right-hand side of the relation should vanish.

For the sufficiency condition consider the infinite dimensional manifold Π of curves $z_2 = f(z_1)$ and all densities $\Psi(z_1)$ on them. Fix $w \in D$. Let $\mu(z_1)$ denote functions having a pole at $z_1 = w_1$, and let $\tilde{\Pi}$ consist of triples (f, Ψ, μ). Let Π_w, $\tilde{\Pi}_w$ denote the sets corresponding to the curves f going through w. Define an integral transformation

$$(5')\qquad \hat{\varphi}(f,\Psi) = \int \varphi(z_1,f(z_1))\,\Psi(z_1)\overline{\Psi(z_1)}\,dz_1\wedge d\overline{z_1} .$$

For functionals $F(f,\Psi)$ on Π define an operator acting into 1-forms on Π_w:

$$(10)\ \varkappa_w F(f,\Psi;\delta f,\delta\Psi) = \delta F(f,\Psi;\ \mu,\delta f,\delta(\Psi\mu)).$$

In other words, the value of the form $\varkappa_w F$ at the point (f,Ψ) on the tangent vector $(\delta f,\delta\Psi)$ equals the value of the variation δF at the same point on the tangent vector $(\mu\delta f,\delta(\Psi\mu))$. The variation δf, being tangent to Π_w, vanishes for $z_1 = w$, and hence the variation $\mu\delta f$ is regular and the vector $(\mu\delta f,\delta(\Psi\mu))$ is tangent to Π (but not to Π_w). Lift the form $\varkappa F$ from Π_w to $\widetilde{\Pi}_w$.

A direct substituion of (5') shows that the form $\varkappa_w\widehat{\varphi}$ is ∂-closed on $\widetilde{\Pi}_w$. The operator $\overline{\varkappa}_w$ is defined in a similar way. Thus one obtain a closed (1,1)-form $(\varkappa_w\wedge\overline{\varkappa}_w)\widehat{\varphi}$ on Π_w.

It remains to note that under condition (i) the form integrated in (7) is a restriction of the form $(\varkappa\wedge\overline{\varkappa})\widehat{\varphi}$ and that under (ii) the cycle γ_w is homological to the corresponding cycle for lines in (2).

4. <u>Locally projective structure on curves of a 2-parameter family</u>. In this section we construct a special class of densities and operators L closely connected with the geometry of the family. We start from the fact that the family Ξ induces on its curves canonical structures of locally projective lines. Namely, for each tangent space $T_\xi\Xi$ consider the projective line $P(T_\xi\Xi)$ of one-dimensional subspaces in it and associate to each point $z\in E_\xi$ the tangent subspace to the curve γ_z at the point ξ. One obtains locally the structure of the projective line on E_ξ.

If the curves E_ξ are defined by equation (3) then the tangent line to γ_z is given by the equation

$$\Phi_1'(z;\xi)\,d\xi_1 + \Phi_2'(z,\xi)\,d\xi_2 = 0,$$

where $\Phi_j' = \partial\Phi/\partial\xi_j$. Hence $\Phi' = \mathrm{grad}_\xi\Phi$ may be considered as homogeneous coordinates of the points on E_ξ with coordinates z: $z\mapsto\Phi'(z;\xi)$.

Introduce a compatible affinization on the curves E_ξ. Fix a curve Λ on D (e.g., z_1 = const). Let $u = u(\xi)$ be the intersection point of E_ξ with Λ. Let

$$(11)\quad t(z;\xi) = \frac{\Phi_1'}{h(\xi)\Phi_1' + \Phi_2'},\quad h(\xi) = -\frac{\Phi_2'(u;\xi)}{\Phi_1'(u;\xi)},$$

i.e. $t(u;\xi)=\infty$. Now one has a fixed affine structure on E_ξ where t is defined up to a linear transformation (with coefficients depending on ξ). Introduce an affine measure on E_ξ

$$(12)\qquad dt(z;\xi) = \Psi(z_1;\xi)\, dz_1,$$

and the auxiliary functions c,d:

$$(13)\quad c(w;\xi) = \lim_{z_1\to w_1} \frac{t(z;\xi) - t(w;\xi)}{z_1 - w_1} \quad ;$$

$$t(z;\xi) - t(w;\xi) = \frac{1}{d(w;\xi)}\, \frac{\Phi_1' + m(w,\xi)\Phi_2'}{n(\xi)\Phi_1' + \Phi_2'}$$

where m is the same as in (8), i.e. $(d\xi_1, d\xi_2) = (d\xi_1, m(w,\xi)d\xi_1)$ is the tangent vector to γ_w at the point ξ and $(d\xi_1, d\xi_2) = (n(\xi)d\xi_2, d\xi_2)$ is the tangent vector to γ_u at the point ξ. Let, finally,

$$(14)\quad L_w = c(w;\xi)\, d(w;\xi)\left(n(\xi)\frac{\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2}\right) c^{-1}(w;\xi)\ .$$

The homogeneous part coincides with derivation along γ_u at the point ξ. The functions c,d give a convenient normalization. One has

$$(15)\quad \mu(z_1;\, w,\xi) = \frac{1}{t(z;\xi) - t(w;\xi)}\quad .$$

We have, therefore, introduced densities Ψ and operators L on Ξ. Condition (ii) is satisfied for them. However, condition (i) is satisfied only under strong additional restrictions imposed on which will be considered in the next section.

One can see without difficulty that all the above constructions are invariant under diffeomorphisms separately with respect to z and to ξ. On the parameter z_1 there depend only normalization of the measure on E_ξ and the corresponding normalization of the measure on γ_w.

5. <u>The infinitesimal Desargues condition</u>. We now impose additional conditions on the family Ξ under which it (considered together with densities and operators L) will be admissible. In that case we call the family itself admissible.

(I.D.) The family Ξ is said to satisfy the infinitesimal Desargues condition if at each point $\xi \in \Xi$ the curves $\gamma_z \ni \xi$ can straightened by a diffeomorphism with respect to ξ in the second order infinitesimal neighbourhood.

Condition (I.D.) is imposed separately at each point ξ.

The main theorem. A family of curves satisfying the infinitesimal Desargues condition is admissible.

Thus if condition (I.D.) is satisifed for Ψ, L constructed in the previous section then formula (7) provides the local inversion formula for the problem of integral geometry. In fact, condition (I.D.) is also a necessary one for the family Ξ to be admissible [1,3].

The main lemma. A family Ξ satisfies the infinitesimal Desargues condition at the point ξ if and only if the curve E_ξ can be transformed to the form $z_2 = 0$ by appropriate diffeomorphisms with respect to z and ξ separately, and if the curves in its neighbourhood satisfy the condition.

$$z_2 = \xi_1 z_1 + \xi_2 + o(|\xi|^2) \tag{16}$$

(where the original curve corresponds to $\xi = 0$ in new parameters).

The sufficiency of (16) is evident. Let us prove the necessaity of that condition which is actually what we need. Let condition (I.D.) be satisfied for $\xi = 0$. Make a diffeomorphism with respect to ξ, which preserves $\xi = 0$ and straightens the curves $\gamma \ni (0,0)$ in the second order neighbourhood of the point $\xi = 0$. Then one has

$$\xi_1 z_1 + \xi_2 + o(|\xi|^2) = 0$$

on $\gamma \ni (0,0)$ where z_1 is taken for a new parameter on the curves $\gamma \ni (0,0)$. For a second new parameter z_2 on the family we take the value of ξ_2 for which γ intersects the axis $\xi_1 = 0$. Then on γ one has $z_2 = 0(|\xi|)$. Suppose now that for a curve γ (close to $\xi = 0$) z_1 is such that the line $\xi_1 z_1 + \xi_2 z_2 = 0$ is tangent to γ at the point $(0, z_2)$. Then one has relation (16) since on γ (z_1, z_2)

$$\xi_1 z_1 + \xi_2 z_2 = Q(z;\xi) + o(|\xi|^2),$$

where Q is a quadratic form in ξ whose coefficients vanish for $z_2 = 0$ hence $Q = o(|\xi|^2)$ because $z_2 = 0(|\xi|)$.

Let us now prove (i). Without any loss of generality (making an appropriate linear diffeomorphism with respect to z) one can assume that Λ is of the form $z_1 = -1/c + o(|\zeta|)$. Consider only the case w = 0. Then

$$t(z;\zeta) = z_1/(cz_1+1) + o(|\zeta|),\ c(0;\zeta) = 1 + o(|\zeta|),$$
$$d(0;\zeta) = 1 + o(|\zeta|),$$
$$m(0;\zeta) = o(|\zeta|),\quad n(\zeta) = c + o(|\zeta|),$$

$$\Psi(z_1;\zeta) = 1/(cz_1+1)^2 + o(|\zeta|),$$

$$L_0 = c\frac{\partial}{\partial\zeta_1} + \frac{\partial}{\partial\zeta_2},\quad \mu(z_1;0,\zeta) = c + 1/z_1 + o(|\zeta|).$$

Now both sides of (i) vanish ends the proof.

We now make some additional comments.

1) Results [1,3] imply that condition (I.D.) is a necessary one for Ξ to be a member of an admissible triple (Ξ, Ψ, L). For Ξ satisfying condition (I.D.) one can effectively describe all admissible triples (Ξ, Ψ, L).

2) Condition (I.D.) can be described in analytical terms in the following way. Introduce arbitrary parameters on γ_z: $\zeta = \zeta(\tau, z)$, $\zeta(0,z) = \zeta^0$. Consider the torsion $\rho(\zeta^0, z) = \dot\zeta_1(0)\ddot\zeta_2(0) - \dot\zeta_2(0)\ddot\zeta_1(0)$. Then condition (I.D.) is equivalent to the fact that ρ is a homogeneous polynomial of the 3rd order in the components of the tangent vector $\dot\zeta(0)$ [1,4]. Using that condition one can extend the family Ξ satisfying condition (I.D.) in such a way that for each direction and each point of a small domain in Ξ there is a unique curve γ going out of it. Accordingly, in the z-representation the curves E_γ are extended to complete rational curves (homeomorphic images of a projective line) $P(T_\zeta\Xi)$.

If there is a complete holomorphic family of curves Ξ such that for each point ζ and for each direction there is a unique curve γ going out of it in that direction, then condition (I.D.) is satisfied automatically. The reason is that $\rho(\zeta^0, z)$ is always a homogeneous function of order 3 in $\dot\zeta(0)$, and since it is a single-valued holomorphic function on $\mathbb{C}^2$ it has to be a polynoamial.

Thus, from the global viewpoint, condition (I.D.) is equivalent to the fact that one has a family of rational curves with the normal bundle $\mathcal{O}(1)$. Roughly speaking, while locally projective structure exists in all cases, in order for the local inversion formula to exist one has to be able to extend that local structure to a global one.

3) the results of [1,5] imply many specific results on addmissible families of curves. For example, such is the family of curves of order 2 on the plane tangent to a triple of fixed curves $\Gamma_1, \Gamma_2, \Gamma_3$. All these are admissible complexes of curves of order 2 in general position. The tangency condition for the curve Γ can be allowed to degenerate into the condition of passing through a fixed point U. A complex of quadrics remains admissible. The complex of quadrics passing through three fixed points U_1, U_2, U_3 is diffeomorphic to the complex of lines on the plane. The set of quadrics going through two fixed points is equivalent to the set of circles. Adding the tangency condition for the curve Γ one obtains an admissible complex. The set of circles of a fixed radius is not admissible.

4) If condition (I.D.) is satisfied both for the complex of curves E_ξ and for the dual family γ_z, then the complex Ξ is diffeomorphic to the family of lines (a necessary and sufficient condition).

6. <u>Integral geometry for complexes of curves on a 3-dimensional manifold</u>. We now extend the above results to the 3-dimensional case. We shall presently see that in that case new features arise which have not been apparent in the 2-dimensional case. However, any subsequent increase in dimension is made automatically.

Suppose that there is a 3-dimensional family of curves E_ξ, $\xi \in \Xi$, in the domain $D \subset \mathbb{C}^2_z$. For each $z \in D$ denote by γ_z the set of such ξ that $z \in E_\xi$. Thus there arises a dual 3-dimensional family of curves γ_z, $z \in D$ on Ξ. One can assume that E_ξ are given by the equations

$$(17) \quad \Phi_1(z,\xi) = 0, \quad \Phi_2(z,\xi) = 0, \quad \xi \in \Xi .$$

As in the 2-dimensional case, fix a set of densities $\psi(z_1;\xi)$ on E_ξ and consider the following integral transformation on $C_0^\infty(D)$

$$(18) \qquad \varphi(z) \mapsto \hat{\varphi}(\xi) = \int_{E_\xi} \varphi(z)\,\psi(z_1;\xi)\,\overline{\psi(z_1;\xi)}\; dz_1 \wedge d\overline{z_1}.$$

We are looking for the local inversion formulas, i.e. when for some operator

$$L_w = L_w^0 + \ell_0(w;\xi), \quad L_w^0 = \ell_1(w;\xi)\frac{\partial}{\partial \xi_1} + \ell_2(w;\xi)\frac{\partial}{\partial \xi_2} + \ell_3(w;\xi)\frac{\partial}{\partial \xi_3}$$

one has

(19) $\varphi(w) = c \int_{\gamma_w} L_w \overline{L_w} \hat{\varphi}(\xi)\, d\xi_1 \wedge d\overline{\xi_1}$, $c - 1/2\pi^2$.

We now formulate sufficient conditions which are in fact also necessary ones [1,3]. We start with the condition whose 2-dimensional analogue is trivial. It ensures the existence of a locally projective structure on the curves of the complex Ξ, which in the 3-dimensional case does not exist in all cases. The local projectivity condition is formulated in the following way:

(L.P.) For each $\xi \in \Xi$ the tangent vectors to the curves $\gamma_z \ni \xi$ $(z \in E_\xi)$ lie in the same 2-dimensional plane $Q(\xi) \subset T_\xi \Xi$. Clearly, in that situation the projective line $P(Q)$ induces a locally projective structure on E_ξ. Our requirement is:

(iii) The operator is a derivation along some direction in $Q(\xi)$.

Condition (L.P.) can also be explained in the following way. The curves $\gamma_z \ni \xi$ span a surface with a conical singularity at ξ; (L.P.) condition then means that the surface is smooth at the point ξ.

Let us reformulate the (L.P.) condition in the z-representation. Vectors of $T_\xi \Xi$ correspond to variations of E_ξ, while vectors tangent to $\gamma_z \ni \xi$, correspond to variations vanishing at some point. The (L.P.) condition means any such variations are proportional with a functional coefficient. Namely, if E_ξ is of the form

(17') $\quad z_j = f_j(z_1; \;\;), \; j = 2,3,$

and

$$M_w = \frac{\partial}{\partial \xi_1} + m_2(w,\xi)\frac{\partial}{\partial \xi_2} + m_3(w;\xi)\frac{\partial}{\partial \xi_3}$$

is a derivation along a tangent vector to γ_w and

$$p_j(z_1; w,\xi) = M_w f_j;$$

$$q_j(z_1; w,\xi) = L_w^0 f_j, \quad j = 2,3,$$

then, since in view of (iii) M_w, L_w^0 derivate along vectors from $Q(\xi)$, there is the following proportionality condition $q_2/p_2 = q_3/p_3 = \mu$

which is exactly (L.P.) condition.

Thus there appeared the function μ which is a major characteristics of the operator L_w under the condition (iii). Conditions (i), (ii) are formulated exactly as in the 2-dimensional case (see section 3).

Theorem. Conditions (i), (ii), (iii) are sufficient (and necessary) for a triple (Ξ, Ψ, L) to be admissible, i.e. for the inversion formula (19) to be true.

The proof of sufficiency repeats that in the 2-dimensional case. On the manifold of triples (f, Ψ, μ) where f is now a vector-function (f_2, f_3) one defines the operator $F \mapsto \varkappa_w F$ where $\varkappa_w$ is of the form (10) and the form $(\varkappa_w \wedge \overline{\varkappa_w})\hat{\varphi}$ is closed. If condition (i) is satisfied then the form integrated in (19) is a restriction of the form $(\varkappa_w \wedge \overline{\varkappa_w})\hat{\varphi}$. Condition (ii) ensures that the cycle γ_w is homological to the cycle of lines passing through w in some plane.

7. 3-dimensional problem (explicit formulae). Our next step is to construct some special Ψ and operators L_w satisfying condition (L.P.) and condition (iii). Let E_ξ be defined by equations (17). Condition (L.P.) means that there exists a combination $\lambda_1 \operatorname{grad}\Phi_1 + \lambda_2 \operatorname{grad}\Phi_2 = (p_1(\xi), p_2(\xi), 1)$ which does not depend on z. However, the coefficients λ_1, λ_2 depend on z. $Q(\xi)$ is given by the equation $p_1(\xi)d\xi_1 + p_2(\xi)d\xi_2 + d\xi_3 = 0$. Substituting $d\xi_3$ into $\langle \operatorname{grad}\Phi_1, d\xi \rangle = 0$ one obtains local projective (homogeneous) coordinates on E_ξ :

$$(20) \quad \begin{aligned} T_1(D;\xi) &= (\Phi_1)'_1 - p_1(\Phi_1)'_3, \\ T_2(z;\xi) &= (\Phi_1)'_2 - p_2(\Phi_1)'_3, \\ (\Phi_1)'_j &= \partial\Phi_1/\partial\xi_j. \end{aligned}$$

Fix an affinizing surface Λ in D and let $u(\xi)$ be the intersection point of Λ with E_ξ. Introduce an affine coordinate

$$t(z;\xi) = \frac{T_1}{n(\xi)T_1 + T_2}, \quad n(\xi) = -\frac{T_2(u;\xi)}{T_1(u;\xi)},$$

so that $t(u;\xi) = 0$. Let

$$dt(z;\xi) = \Psi(z_1;\xi)dz_1,$$

$$c(w;\xi) = \lim_{z_1 \to w_1} \frac{t(z;\xi) - t(w;\xi)}{z_1 - w_1},$$

$$t(z;\xi) - t(w;\xi) = \frac{1}{d(w;\xi)} \frac{R_1 + m(w;\xi)T_2}{n(\xi)T_1 + T_2}, \qquad (21)$$

$$L_w = c(w,\xi)d(w,\xi)(n(\xi)\frac{\partial}{\partial\xi_1} + \frac{\partial}{\partial\xi_2} - (n(\xi)p_1(\xi)+p_2(\xi))\frac{\partial}{\partial\xi_3})c^{-1}(w,\xi).$$

Note that the tangent vector to γ_w at the point ξ is of the form

$$(d\xi_1, d\xi_2, d\xi_3) = (d\xi_1, m(w,\xi)d\xi_1, (-p_1-m(w,\xi)p_2)d\xi_1)$$

and

$$\mu(z_1; w,\xi) = \frac{1}{t(z;\xi) - t(w;\xi)} .$$

The infinitesimal Desargues condition (I.D.) is formulated as in the 2-dimensional case: $\gamma \ni \xi$ can be straightened in the second infinitesimal neighbourhood.

Theorem. *A complex of curves Ξ satisfying conditions (L.P.) and (I.D.) is admissible, i.e. the triple (Ξ, Ψ, L) where Ψ, L are defined by (21) is admissible and the inversion formula (19) holds.*

Necessity of those conditions follow from [1,2].

Lemma. If conditions (L.P.) and (I.D.) are satisfied then any curve in a neighbourhood of each curve E_ξ can by appropriate diffeomermisms with respect to z, ξ be reduced to the form

$$(22) \qquad \begin{aligned} z_2 &= \xi_1 z_1 + \xi_2 + o(|\xi|^2), \\ z_3 &= \xi_3 + o(|\xi|^2), \end{aligned}$$

where E_ξ corresponds to $\xi = 0$.

The proof of the Lemma repeats that in the 2-dimensional case. Define a transformation in a neighbourhood of E_ξ, $\xi = 0$. First, choose such ξ_3 in the neighbourhood that all γ passing through the point $(0,0,z_3)$ satisfy the condition $\xi_3 - z_3 = o(|\xi_3|^2)$. That is possible since under the condition (L.P.) one has

$$0=\sum p_j(\eta)(\xi_j-\eta_j) + \frac{1}{2}\sum_{i<j}(p'_{i,j}(\eta)+p'_{j,i}(\eta))(\xi_i-\eta_i)(\xi_j-\eta_j)+o(|\xi-\eta|^2),$$

$$p'_{i,j} = \partial p_i/\partial\xi_j.$$

So far we have used only the (L.P.) condition; z_3 is one of the parameters. Now using (I.D.) we make such a diffeomorphism with respect to ξ that on $\gamma\ni(0,0,z_3)$ one has $\xi_1 z_1+\xi_2 = o(|\xi|^2)$. Finally, we choose z_1, z_2 in such a way that the curves $\gamma(z_1,z_2,z_3)$ pass through $\gamma\ni(0,z_2,z_3)$ and are tangent to the line $\{\xi_1 z_1+\xi_2 z_2=0, \xi_3 = z_3\}$. The validity of (22) is established as in the 2-dimensional case.

Admissibility of Ξ is verified with the use of (22) exactly as in the 2-dimensional case (all the formulas being identical).

All the comments made on the 2-dimensional admissible complexes of curves are naturally transferred to the 3-dimensional case. The curves of an admissible complex are canonically extended to rational curves (with the normal bundle $\mathcal{O}(1) + \mathcal{O}$. Under condition (L.P.) the torsion is defined as the vector product $\rho = [\ddot{\xi}(0), \dot{\xi}(0)]$. Condition (I.D.) means that ρ is a polynomial of the 3rd order in $\dot{\xi}(0)$, and it is sufficient to require that only one of the components is polynomial [6]. For a globally holomorphic family conditions (L.P.) and (I.D.) mean that for each point ξ and each direction from some plane $Q(\xi)$ there is a unique curve γ going out of that point in that direction.

As for examples of admissible complexes of curves, an untrivial question is already that on the admissible complexes of lines. In that case, evidently, there remains only the condition (L.P.) Admissible complexes in general position consist of lines tangent to a fixed surface S or intersecting some curve Γ[2]. In the 8-parameter family of curves of order 2 in $\mathbb{C}P^3$ admissible complexes are defined by 5 conditions: either those of being tangent to a fixed surface S or those of intersecting a fixed curve Γ [1,5].

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