Jacek Gancarzewicz; Modesto R. Salgado

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HORIZONTAL LIFTS OF TENSOR FIELDS AND CONNECTIONS TO THE TANGENT BUNDLE OF HIGHER ORDER

Jacek Gancarzewicz and Modesto Salgado

0. Introduction

Let $T^rM = \{j_0^r\gamma \mid \gamma:(-\varepsilon,+\varepsilon) \longrightarrow M \text{ of class } C^\infty\}$ be the tangent bundle of order r, where M is a manifold of dimension n. We denote by

$$\pi : T^rM \longrightarrow M \quad , \quad \pi(j_0^r\gamma) = \gamma(0)$$

the bundle projection. Let Γ be a connection of order r on M, that is, Γ is a connection in the principal fibre bundle F^rM of frames of order r. Since T^rM is an associated bundle with F^rM, this connection defines a distribution H on T^rM, called the horizontal distribution, such that

$$T(T^rM) = V(T^rM) \oplus H$$

where $V(T^rM) = \ker d\pi$ is the distribution of vertical vectors on T^rM.

The restriction $d_p\pi|H_p$ is an isomorphism of H_p onto $T_{\pi(p)}M$ and we can define the horizontal lift X^H of a vector field X from M to T^rM by the formula

$$X^H(p) = (d_p\pi|H_p)^{-1}(X_{\pi(p)})$$

In this paper we will discuss horizontal lifts of tensor fields from M to T^rM.

This paper has six sections.

In Section 1 we recall results of A. Morimoto [7], [11] about lifts of tensor fields to the bundle T^rM.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In Section 2 we study horizontal lifts of vector fields and 1-forms. For every $\nu = 0,\dots,r$ we define the horizontal ν-lift of 1-forms from M to T^rM and we study properties of

$$\gamma^{\nu}(\omega) = \omega^{(\nu)} - \omega^{H,\nu}$$

where ω is an 1-form on M and $\omega^{(\nu)}$, $\omega^{H,\nu}$ denote, respectively, the ν-lift and the horizontal ν-lift of ω to T^rM. We have

$$\gamma^{\nu}(f\omega) = \sum_{\mu=0}^{\nu} f^{(\mu)} \gamma^{\nu-\mu}(\omega)$$

for every function f on M and every 1-form ω on M.

Since for a vector field X on M and

$$\gamma^{r}(X) = X^{(r)} - X^{H}$$

we have

$$\gamma^{r}(fX) = f^{(0)} \gamma^{(r)}(X) + \sum_{\nu=1}^{r} f^{(\nu)} X^{(r-\nu)}$$

then we obtain

$$\gamma^{r}(fX) = \sum_{\nu=0}^{r} f^{(\nu)} \gamma^{(r-\nu)}(X)$$

if we define

$$\gamma^{\nu}(X) = X^{(\nu)}$$

for $\nu \leq r-1$.

In Section 3 using the methods of A. Morimoto we find the prolongation of the operation $\gamma^{\nu}(t)$ fot any tensor field t on M and we define the horizontal ν-lift of t from M to T^rM by the formula

$$t^{H,\nu} = t^{(\nu)} - \gamma^{\nu}(t)$$

and we study the principal properties of these operations for $\nu = 1, \dots, r$. (The horizontal 0-lift is not interesting because $t^{H,0} \equiv 0$ for any tensor field t on M.)

The proposed definition gives a generalization of known cases. If $r = 1$, $t^{H,1}$ coincides with the horizontal lift defined by K. Yano and S. Ishihara [12]. If r is arbitrary and t is a tensor field of type (1,1), then $t^{H,r}$ coincides with the horizontal lift t^H defined by J. Gancarzewicz, S. Mahi and N. Rahmani [5].

In Section 4 we study horizontal lifts of pseudoriemannian metrics.

In Section 5 we define a horizontal lift of linear connection ∇ from M to T^rM with respect to a given connection Γ of order r. Thus, for r = 1, we have a horizontal lift of ∇ (from M to TM) with respect to a linear connection ∇_0. If $\nabla = \nabla_0$ then this operation coincides with the horizontal lift of linear connections introduced by K. Yano and S. Ishihara [12].

In Section 6 we stutd the relationship between the horizontal lifts of tensor fields and linear connections.

In this paper all manifolds are differentiable of class C^∞ and all objects (as functions, vector fields, forms, tensor fields etc.) are always of class C^∞ .

1. Lifts of tensor fields to the tangent bundle of higher order

In the first section we recall briefly the main results of A. Morimoto [7], [11] about lifts of tensor fields to the tangent bundle of higher order. These results will be used in the sequel.

Let us denote by T^rM the bundle of r-jets at 0 of curves γ of class C^∞ on a manifold M. If f is a function on M and $\nu = 0,\dots,r$ we define the ν-lift $f^{(\nu)}$ as the function on T^rM given by the formula

$$f^{(\nu)}(j_0^r\gamma) = \frac{1}{\nu!}\frac{d^\nu}{dt^\nu}(f\circ\gamma)(0) \quad . \tag{1.1}$$

If ν is negative, then we set $f^{(\nu)} = 0$.

For a chart (U,x^i) on M we consider the induced chart $(\pi^{-1}(U),x^{i,\nu})$ on T^rM defined by

$$x^{i,\nu} = (x^i)^{(\nu)} \tag{1.2}$$

The family of ν-lifts of functions is very important because, if $\widetilde{X}$ and $\widetilde{Y}$ are vector fields on T^rM such that $\widetilde{X}(f^{(\nu)}) = \widetilde{Y}(f^{(\nu)})$ for every function f on M and every $\nu = 0,\dots,r$, then $\widetilde{X} = \widetilde{Y}$.

If X is a vector field on M and $\nu = 0,\dots,r$, then there is one and only one vector field $X^{(\nu)}$ on T^rM such that

$$X^{(\nu)}(f^{(\lambda)}) = (Xf)^{(\nu+\lambda-r)} \tag{1.3}$$

for all functions f on M and $\lambda = 0,\dots,r$ (see [7], [11]). The vector field $X^{(\nu)}$ on T^rM is called the ν-lift of X.

Formulas (1.2) and (1.3) imply

$$(1.4) \qquad \frac{\partial}{\partial x^{i,\nu}} = \left(\frac{\partial}{\partial x^i}\right)^{(r-\nu)}$$

for $\nu = 0,\dots,r$ and $i = 1,\dots,n$ ($n = \dim M$).

If ω is an 1-form on M and $\nu = 0,\dots,r$, then there is one and only one 1-form $\omega^{(\nu)}$ on T^rM such that

$$(1.5) \qquad \omega^{(\nu)}(X^{(\lambda)}) = (\omega X)^{(\nu+\lambda-r)}$$

for all vector fields X on M and $\lambda = 0,\dots,r$. The 1-form $\omega^{(\nu)}$ on T^rM is called the ν-lift of ω (see [7], [11]).

From formulas (1.2) - (1.5) we have

$$(1.6) \qquad dx^{i,\nu} = (dx^i)^{(\nu)}$$

for $\nu = 0,\dots,r$ and $i = 1,\dots,n$. Using formulas (1.1), (1.3) and (1.5) we can verify (see [7], [11]) the following properties of ν-lifts

$$(1.7) \qquad (f+g)^{(\nu)} = f^{(\nu)} + g^{(\nu)} \quad , \quad (fg)^{(\nu)} = \sum_{\mu=0}^{\nu} f^{(\mu)}\, g^{(\nu-\mu)}$$

$$(1.8) \qquad (X+Y)^{(\nu)} = X^{(\nu)} + Y^{(\nu)} \quad , \quad (fX)^{(\nu)} = \sum_{\mu=0}^{\nu} f^{(\mu)}\, X^{(\nu-\mu)}$$

$$(1.9) \qquad (\omega+\tau)^{(\nu)} = \omega^{(\nu)} + \tau^{(\nu)} \quad , \quad (f\omega)^{(\nu)} = \sum_{\mu=0}^{\nu} f^{(\mu)} \quad {}^{(\nu-\mu)}$$

for all functions f, g, all vector fields X, Y, all 1-forms ω, τ and $\nu = 0,\dots,r$. From (1.3) we also obtain

$$(1.10) \qquad [X^{(\nu)}, Y^{(\mu)}] = [X,Y]^{(\nu+\mu-r)}$$

Using formulas (1.7), (1.8) and (1.9) we can prove (see A. Morimoto [7], [11]).

Proposition 1.1. For each $\nu = 0,\dots,r$ there is one and only one operation $t \longrightarrow t^{(\nu)}$ which transforms tensor fields on M into tensor fields on T^rM and satisfies the following conditions:

(a) If t is of type (p,q) on M, then $t^{(\nu)}$ is of type (p,q) on T^rM.

(b) If t is of type (0,0) (respectively, of type (1,0) and (0,1)), then $t^{(\nu)}$ is given by formula (1.1) (respectively, by (1.3) and

(1.5)).

(c) The operation $t \longrightarrow t^{(\nu)}$ is linear with respect to constant coefficients.

(d) For two tensor fields t and t' on M we have

$$(t \otimes t')^{(\nu)} = \sum_{\mu=0}^{\nu} t^{(\mu)} \otimes t'^{(\nu-\mu)} \tag{1.11}$$

The tensor field $t^{(\nu)}$ is called the ν-lift of t to T^rM. From (1.11), by induction, we have

$$(t_1 \otimes \dots \otimes t_p)^{(\nu)} = \sum_{\nu_1+\dots+\nu_p=\nu} t_1^{(\nu_1)} \otimes \dots \otimes t_p^{(\nu_p)} \tag{1.12}$$

for every tensor fields $t_1,\dots,t_p$ on M.

Using the above proposition and formulas (1.1), (1.3) and (1.5), for a tensor t of type (0,p) or (1,p), we can obtain (see also [7], [11])

$$t^{(\nu)}(X_1^{(\mu_1)},\dots,X_p^{(\mu_p)}) = (t(X_1,\dots,X_p))^{(\nu+\mu_1+\dots+\mu_p-rp)} \tag{1.13}$$

for all vector fields $X_1,\dots,X_p$ on M and $\nu,\mu_1,\dots,\mu_p = 0,\dots,r$; where on the right-hand side of formula (1.13) we have the $(\nu+\mu_1+\dots+\mu_p-pr)$-lift of a function $t(X_1,\dots,X_p)$ in the case of a tensor field of type (0,p) or of a vector field in the case of a tensor field of type (1.p).

Using formulas (1.2), (1.4), (1.6) and (1.12) for

$$t = t^{i_1\dots i_p}_{j_1\dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

we can find the local expression of $t^{(\nu)}$. Namely, we have (see [11])

$$t^{(\nu)} = \sum_{\substack{\nu_1,\dots,\nu_p \\ \eta_1,\dots,\eta_q}} (t^{i_1\dots i_p}_{j_1\dots j_q})^{(\nu-\Sigma\nu_i-\Sigma\eta_j+rp)} \frac{\partial}{\partial x^{i_1,\nu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p,\nu_p}} \otimes dx^{j_1,\eta_1} \otimes \dots \otimes dx^{j_q,\eta_q} \tag{1.14}$$

2. Horizontal lifts of vector fields and 1-forms

Let Γ be a connection of order r on M. For each point y of T^rM,

Γ defines a horizontal subspace H_y of $T_y(T^rM)$ such that

$$T_y(T^rM) = H_y \oplus V_y(T^rM)$$

where $V_y(T^rM) = \ker d_y\pi$ is the vertical space at y. Since $d_y\pi|H_y$ is an isomorphism of H_y onto $T_{\pi(y)}M$, we can define the horizontal lift X^H of a vector field X from M to T^rM by the formula

$$X^H(y) = (d_y\pi|H_y)^{-1}(X_{\pi(y)}) \quad .$$

If (U,x^i) is a chart on M and

$$X = X^i \frac{\partial}{\partial x^i} \quad , \qquad X^H = X^{i,\nu} \frac{\partial}{\partial x^{i,\nu}}$$

then we have (see [5])

$$X^{i,0} = X^i$$

$$X^{i,\nu} = -\sum_{s=1}^{\nu} \frac{1}{s!} \sum_{\substack{\nu_1+\dots+\nu_s=\nu \\ \nu_1,\dots,\nu_s>0}} X^j \Gamma^i_{ji_1\dots i_s} x^{i_1,\nu_1} \dots x^{i_s,\nu_s} \tag{2.1}$$

for $\nu = 1,\dots,r$ and $i = 1,\dots,n$, where $\Gamma^i_{ji_1\dots i_s}$ are the components of Γ.

We propose the following definition of horizontal ν-lift of 1-forms from M to T^rM.

Definition 2.1. Let ω be an 1-form on M and $\nu = 0,\dots,r$. We define the 1-form $\omega^{H,\nu}$ on T^rM by the formulas

$$\omega^{H,\nu}(X^H) = 0$$

$$\omega^{H,\nu}(X^{(\lambda)}) = (\omega(X))^{(\nu+\lambda-r)} \tag{2.2}$$

for all vector fields X on M and $\lambda = 0,\dots,r-1$.

The 1-form $\omega^{H,\nu}$ is called the horizontal ν-lift of ω from M to T^rM with respect to the given connection Γ of order r.

Let us note that $\omega^{H,\nu}$ is a well-defined 1-form on T^rM and the restrictions of $\omega^{H,\nu}$ and $\omega^{(\nu)}$ to the vertical space $V_y(T^rM)$ coincide. Also, if $\nu = 0$, then $\omega^{H,0} = 0$ on T^rM.

In the case $r = 1$, $\omega^{H,1}$ coincides with the horizontal lift ω^H defined by K. Yano and S. Ishihara [12], [13].

If (U,x^i) is a chart on M and

$$\omega = \omega_i\, dx^i \quad , \quad \omega^{H,\nu} = \omega^{H,\nu}_{i,\mu}\, dx^{i,\mu}$$

then from (1.4, (1.6), (2.1) and (2.2) we have

$$\omega^{H,\nu}_{i,0} = \sum_{\lambda=1}^{\nu} \sum_{s=1}^{r} \frac{1}{s!} \sum_{\substack{\lambda_1+\dots+\lambda_s=\lambda \\ \lambda_1,\dots,\lambda_s>0}} \Gamma^{j}_{ii_1\dots i_s}\, x^{i_1,\lambda_1} \dots x^{i_s,\lambda_s} (\omega_j)^{(\nu-\lambda)} \tag{2.3}$$

$$\omega^{H,\nu}_{i,\mu} = (\omega_i)^{(\nu-\mu)}$$

for $\mu = 1,\dots,r$.

The horizontal ν-lift of 1-forms has the following elementary properties.

<u>Proposition 2.2</u>. If ω, ω' are 1-forms on M, f is a function on M and $\nu = 0,\dots,r$, then

$$(\omega + \omega')^{H,\nu} = \omega^{H,\nu} + \omega'^{H,\nu} \tag{2.4}$$

$$(f\omega)^{H,\nu} = \sum_{\mu=0}^{\nu} f^{(\mu)}\, \omega^{H,\nu-\mu} \tag{2.5}$$

<u>Proof</u>. The first formula is trivial. To show the second formula we observe that for every vector field X on M we have

$$\Big(\sum_{\mu=0}^{\nu} f^{(\mu)}\, \omega^{H,\nu-\mu}\Big)(X^H) = 0 = (f\omega)^{H,\nu}(X^H)$$

and for $\lambda = 0,\dots,r-1$

$$\begin{aligned}\Big(\sum_{\mu=0}^{\nu} f^{(\mu)}\, \omega^{H,\nu-\mu}\Big)(X^{(\lambda)}) &= \sum_{\mu=0}^{\nu} f^{(\mu)}\, (\omega(X))^{(\nu-\mu+\lambda-r)} \\ &= \sum_{\mu=0}^{\nu+\lambda-r} f^{(\mu)}\, (\omega(X))^{(\nu-\mu+\lambda-r)} \\ &= (f\omega(X))^{(\nu+\lambda=r)} \\ &= (f\omega)^{H,\nu}(X^{(\lambda)})\end{aligned}$$

since for $\mu > \nu+\lambda-r$ and $\lambda \leq r-1$, $\nu-\mu+\lambda-r$ is negative and hence $(\omega(X))^{(\nu-\mu+\lambda-r)} \equiv 0$, and the results follows.

If ω is an 1-form on M, we define

$$\gamma^{\nu}(\omega) = \omega^{(\nu)} - \omega^{H,\nu} \tag{2.6}$$

So, $\gamma^{\nu}(\omega)$ is an 1-form on T^rM which measures the deformation between the ν-lift and the horizontal ν-lift of ω.

Now from (1.9) and Proposition 2.2 we deduce

$$\gamma^{\nu}(\omega + \omega') = \gamma^{\nu}(\omega) + \gamma^{\nu}(\omega') \tag{2.7}$$

$$\gamma^{\nu}(f\omega) = \sum_{\mu=0}^{\nu} f^{(\mu)} \gamma^{\nu-\mu}(\omega) \tag{2.8}$$

for all 1-forms ω, ω' on M and all functions f on M.

If X is a vector field on M, we define

$$\gamma^{r}(X) = X^{(r)} - X^{H} \tag{2.9}$$

So, $\gamma^r(X)$ is a vector field on T^rM which measures the deformation between the complete lift and the horizontal lift of X.

Using the formula

$$(fX)^H = f^{(0)} X^H$$

and (1.8) we obtain

$$\begin{aligned} \gamma^r(fX) &= (fX)^{(r)} - (fX)^H \\ &= \sum_{\mu=0}^{r} f^{(\mu)} X^{(r-\mu)} - f^{(0)} X^H \\ &= f^{(0)} \gamma^r(X) + \sum_{\mu=1}^{r} f^{(\mu)} X^{(r-\mu)} \end{aligned}$$

If we want to have an analogous formula to (2.8) for vector fields in case $\nu = r$, we must define

$$\gamma^{\mu}(X) = X^{(\mu)} \tag{2.10}$$

for $\mu = 0,\dots,r-1$.

Now we have

$$\gamma^{r}(X) = \sum_{\mu=0}^{r} f^{(\mu)} \gamma^{r-\mu}(X) \tag{2.11}$$

Using formulas (1.8) and (2.8) - (2.10) it is easy to verify

Proposition 2.3. If X, X' are vector fields on M, f is a function on M and $\nu = 0,\dots,r$, then

$$\gamma^{\nu}(X + X') = \gamma^{\nu}(X) + \gamma^{\nu}(X') \tag{2.12}$$

$$\gamma^{\nu}(fX) = \sum_{\mu=0}^{\nu} f^{(\mu)} \gamma^{\nu-\mu}(X) \tag{2.13}$$

3. Horizontal lifts of tensor fields

In Section 2 we defined the operations γ^{ν}, $\nu = 0,\dots,r$, for vector fields and 1-forms. If we denote $\gamma^{\nu}(f) = f^{(\nu)}$ for any function f on M, then Propositions 2.2 and 2.3 imply

$$\begin{aligned} \gamma^{\nu}(fX) &= \sum_{\mu=0}^{\nu} \gamma^{\mu}(f)\, \gamma^{\nu-\mu}(X) \\ \gamma^{\nu}(f\omega) &= \sum_{\mu=0}^{\nu} \gamma^{\mu}(f)\, \gamma^{\nu-\mu}(\omega) \\ \gamma^{\nu}(X + X') &= \gamma^{\nu}(X) + \gamma^{\nu}(X') \\ \gamma^{\nu}(\omega + \omega') &= \gamma^{\nu}(\omega) + \gamma^{\nu}(\omega') \end{aligned} \tag{3.1}$$

Now, using the same arguments as A. Morimoto in the proof of Proposition 3.1 in [11] we can prolonge the operations γ^{ν}, $\nu = 0,\dots, r$, for any tensor fields. We have the following proposition.

Proposition 3.1. Let $\mathcal{T}(M)$ denote the algebra of tensor fields on M. For any $\nu = 0,\dots,r$, there is one and only one operation

$$\gamma^{\nu} : \mathcal{T}(M) \longrightarrow \mathcal{T}(T^{r}M)$$

such that

(a) If t is a tensor field of type (p,q) on M, then $\gamma^{\nu}(t)$ is a tensor field of type (p,q) on $T^{r}M$.

(b) If t and t' are tensor fields on M, then we have

$$\begin{aligned} \gamma^{\nu}(t + t') &= \gamma^{\nu}(t) + \gamma^{\nu}(t') \\ \gamma^{\nu}(t \otimes t') &= \sum_{\mu=0}^{\nu} \gamma^{\mu}(t) \otimes \gamma^{\nu-\mu}(t') \end{aligned}$$

(c) If X is a vector field on M, then

$$\gamma^{\nu}(X) = \begin{cases} X^{(r)} - X^{H} & \text{if } \nu = r \\ X^{(\ )} & \text{if } \nu < r \end{cases}$$

(d) If ω is an 1-form on M, then

$$\gamma^{\nu}(\omega) = \omega^{(\nu)} - \omega^{H,\nu}$$

where $\omega^{H,\nu}$ is the horizontal ν-lift of ω defined by (2.2).

(e) If f is a function on M, then $\gamma^{\nu}(f) = f^{(\nu)}$.

From (b) we easily obtain by induction

$$\gamma^{\nu}(t_1 \otimes \ldots \otimes t_p) = \sum_{\nu_1+\ldots+\nu_p=\nu} \gamma^{\nu_1}(t_1) \otimes \ldots \otimes \gamma^{\nu_p}(t_p) \tag{3.2}$$

where $t_1, \ldots t_p$ are tensor fields on M. Next we look for explicit formulas for $\gamma^{\nu}(t)$, where t is a tensor field of some special types.

<u>Proposition 3.2</u>. If $\nu = 0,\ldots,r$ and t is a tensor field of type (0,p) on M, then

(a) $$\gamma^{\nu}(t)(X_1^H,\ldots,X_p^H) = t^{(\nu)}(X_1^H,\ldots,X_p^H)$$

for all vector fields $X_1,\ldots,X_p$ on M.

(b) If there is a vertical vector among $\tilde{X}_1,\ldots,\tilde{X}_p \in T_{y_0}(T^rM)$, then

$$\gamma^{\nu}(t)(\tilde{X}_1,\ldots,\tilde{X}_p) = 0$$

<u>Proof</u>. Fistly, we suppose that $t = \omega_1 \otimes \ldots \otimes \omega_p$, where $\omega_1,\ldots,\omega_p$ are 1-forms on M. Now, according to (3.2) we have

$$\gamma^{\nu}(t) = \sum_{\nu_1+\ldots+\nu_p=\nu} \gamma^{\nu_1}(\omega_1) \otimes \ldots \otimes \gamma^{\nu_p}(\omega_p) \tag{3.3}$$

From Proposition 3.1(d) we obtain

$$\gamma^{\mu}(\omega_i)(X_i^H) = \omega_i^{(\mu)}(X_i^H)$$

and hence

$$\gamma^{\nu}(t)(X_1^H,\ldots,X_p^H) = \sum_{\nu_1+\ldots+\nu_p=\nu} \omega_1^{(\nu_1)}(X_1^H)\ldots\omega_p^{(\nu_p)}(X_p^H)$$

$$= t^{(\nu)}(X_1^H,\ldots,X_p^H)$$

To prove the second formula for $t = \omega_1 \otimes \dots \otimes \omega_p$, let $\tilde{X}_1, \dots, \tilde{X}_p$ be vectors tangent at $y_o \in T^rM$ to T^rM such that for some i_o, $1 \leq i_o \leq p$, X_{i_o} is vertical. There are a vector field Y on M and λ, $0 \leq \lambda \leq r-1$, such that $X_{i_o} = Y^{(\lambda)}(y_o)$. Now, according to (3.3) we have

$$\gamma^{\nu}(t)(\tilde{X}_1, \dots, \tilde{X}_p) = \sum_{\nu_1 + \dots + \nu_p = \nu} \gamma^{\nu_1}(\omega_1)(\tilde{X}_1) \dots \gamma^{\nu_p}(\omega_p)(\tilde{X}_p) = 0$$

because from (1.5), (2.2) and (2.6) we obtain

$$\begin{aligned} \gamma^{\nu_{i_o}}(\omega_{i_o})(\tilde{X}_{i_o}) &= \gamma^{\nu_{i_o}}(\omega_{i_o})(Y^{(\lambda)}(y_o)) \\ &= (\omega_{i_o}^{(\nu_{i_o})}(Y^{(\lambda)}))(y_o) - (\omega^{H,\nu_{i_o}}(Y^{(\lambda)}))(y_o) \\ &= 0 \ . \end{aligned}$$

Let $\mathcal{K}$ be the family of tensor fields t of type (0,p) on M such that t verify Proposition 3.2. We proved that tensor fields of type $\omega_1 \otimes \dots \otimes \omega_p$ belong to $\mathcal{K}$, where $\omega_1, \dots, \omega_p$ are 1-forms on M. From the linearity of γ^{ν} and ν-lifts we obtain that if t and t' belong to $\mathcal{K}$ then so is t + t'. Since every tensor field of type (0,p) is a sum of tensor fields of type $\omega_1 \otimes \dots \otimes \omega_p$, $\mathcal{K}$ contains all tensor fields of type (0,p), and the proof is complete.

To prove the analogous proposition for tensor fields of type (1,p) we will need the following lemma.

<u>Lemma 3.3</u>. If g is a tensor field of type (0,p) on M and $X_1, \dots, X_p$ are vector fields on M, then

$$g^{(0)}(X_1^H, \dots, X_p^H) = (g(X_1, \dots, X_p))^{(0)}$$

<u>Proof</u>. It is trivial from the definition of the 0-lift of tensors and formula (2.1).

Now we can prove the following proposition.

<u>Proposition 3.4</u>. If t is a tensor field of type (1,p) on M, $p > 0$, and $\nu = 0, \dots, r$, then

(a) If $X_1, \dots, X_p$ are vector fields on M, then we have

$$\gamma^{\nu}(t)(X_1^H, \dots, X_p^H) = \begin{cases} t^{(\nu)}(X_1^H, \dots, X_p^H) & \text{if } \nu < r \\ t^{(r)}(X_1^H, \dots, X_p^H) - (t(X_1, \dots, X_p))^H & \text{if } \nu = r \end{cases}$$

(b) If there is a vertical vector among $\tilde{X}_1,\dots,\tilde{X}_p \in T_{y_0}(T^rM)$, then

$$\gamma^\nu(t)(\tilde{X}_1,\dots,\tilde{X}_p) = 0$$

<u>Proof</u>. We suppose, at first, that $t = g \otimes X$, where X is a vector field on M and g is a tensor field of type $(0,p)$ on M. From Proposition 3.1(b) we have

$$\gamma^\nu(t) = \sum_{\mu=0}^{\nu}{}' \gamma^\nu(g) \otimes \gamma^{\nu-\mu}(X) \tag{3.4}$$

Now, for vector fields $X_1,\dots,X_p$ on M, according to Proposition 3.2, we have

$$\begin{aligned}\gamma^\nu(t)(X_1^H,\dots,X_p^H) &= \sum_{\mu=0}^{\nu} \gamma^\mu(g)(X_1^H,\dots,X_p^H)\ \gamma^{\nu-\mu}(X)\\ &= \sum_{\mu=0}^{\nu} g^{(\mu)}(X_1^H,\dots,X_p^H)\ \gamma^{\nu-\mu}(X)\end{aligned}$$

and next, using (2.9), (2.10) and Proposition 3.2, we obtain

$$\gamma^\nu(t)(X_1^H,\dots,X_p^H) = \begin{cases} \displaystyle\sum_{\mu=0}^{\nu} g^{(\mu)}(X_1^H,\dots,X_p^H)\ X^{(\nu-\mu)} & \text{if } \nu < r \\ \displaystyle\sum_{\mu=0}^{\nu} g^{(\mu)}(X_1^H,\dots,X_p^H)\ X^{(\nu-\mu)} & \\ \qquad - g^{(0)}(X_1^H,\dots,X_p^H)\ X^H & \text{if } \nu = r \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{\mu=0}^{\nu} (g^{(\mu)} \otimes X^{(\nu-\mu)})(X_1^H,\dots,X_p^H) & \text{if } \nu < r \\ \displaystyle\sum_{\mu=0}^{\nu} (g^{(\mu)} \otimes X^{(\nu-\mu)})(X_1^H,\dots,X_p^H) & \\ \qquad - (g(X_1,\dots,X_p))^{(0)}\ X^H & \text{if } \nu = r \end{cases}$$

From Proposition 1.1, for every $\nu = 0,\dots,r$, we know that

$$\sum_{\mu=0}^{\nu} g^{(\mu)} \otimes X^{(\nu-\mu)} = (g \otimes X)^{(\nu)} = t^{(\nu)}$$

and, on the other hand, from Lemma 3.3

$$\begin{aligned}(g(X_1,\dots,X_p))^{(0)}\ X^H &= (g(X_1,\dots,X_p)\ X)^H = ((g\otimes X)(X_1,\dots,X_p))^H\\ &= (t(X_1,\dots,X_p))^H .\end{aligned}$$

The above remarks finish the proof of part (a) of the proposition for a tensor field $t = g \otimes X$. Part (b) of Proposition 3.4 for $t = g \otimes X$ is an immediate consequence of Proposition 3.2(b). Using the same arguments as in the end of the proof of Proposition 3.2 we can prove that Proposition 3.4 is true for all tensor fields of type $(1,p)$, $p > 0$.

Now we propose the following definition of horizontal ν-lift of tensor fields.

Definition 3.5. Let t be a tensor field of type (p,q) on M and $\nu = 0,\dots,r$. The tensor field (of type (p,q))

$$t^{H,\nu} = t^{(\nu)} - \gamma^{\nu}(t)$$

on T^rM is called the horizontal ν-lift of t from M to T^rM with respect to a given connection of order r on M.

To finish this section we give a few remarks.

Remark 3.6. The horizontal 0-lift is not interesting because for every tensor field t, $t^{H,0} \equiv 0$. In fact, if

$$t = t^{i_1\dots i_p}_{j_1\dots j_q}\frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

is a tensor field of type (p,q) on M, then from formulas (1.12) and (3.2) we have

$$\begin{aligned}\gamma^0(t) &= \gamma^0(t^{i_1\dots i_p}_{j_1\dots j_q})\,\gamma^0\left(\frac{\partial}{\partial x^{i_1}}\right) \otimes \dots \otimes \gamma^0\left(\frac{\partial}{\partial x^{i_p}}\right) \\ &\qquad \otimes \gamma^0(dx^{j_1}) \otimes \dots \otimes \gamma^0(dx^{j_q}) \\ &= (t^{i_1\dots i_p}_{j_1\dots j_q})^{(0)} \left(\frac{\partial}{\partial x^{i_1}}\right)^{(0)} \otimes \dots \otimes \left(\frac{\partial}{\partial x^{i_p}}\right)^{(0)} \\ &\qquad \otimes (dx^{j_1})^{(0)} \otimes \dots \otimes (dx^{j_q})^{(0)} \\ &= t^{(0)}\end{aligned}$$

and hence

$$t^{H,0} = t^{(0)} - \gamma^0(t) \equiv 0$$

Remark 3.7. If t is of type $(0,1)$ on M, then $t^{H,\nu}$ is given by formula (2.2). If t is of type $(1,0)$, then $t^{H,r} = t^H$ is defined in Section 2 and $t^{H,\nu} \equiv 0$ for $\nu < r$.

Remark 3.8. Since $\gamma^{\nu}(X) = X^{(\nu)}$ for $\nu < r$ and all vector fields X on M, if

$$t = t^{i_1\dots i_p} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}}$$

is a tensor field of type (p,0), the for $\nu < r$ we have

$$\gamma^{\nu}(t) = \sum_{\mu_0+\dots+\mu_p=\nu} \gamma^{\mu_0}(t^{i_1\dots i_p})\, \gamma^{\mu_1}(\frac{\partial}{\partial x^{i_1}}) \otimes \dots \otimes \gamma^{\mu_p}(\frac{\partial}{\partial x^{i_p}})$$

$$= \sum_{\mu_0+\dots+\mu_p=\nu} (t^{i_1\dots i_p})^{(\mu_0)} (\frac{\partial}{\partial x^{i_1}})^{(\mu_1)} \otimes \dots \otimes (\frac{\partial}{\partial x^{i_p}})^{(\mu_p)}$$

$$= t^{(\nu)}$$

and hence, $t^{H,\nu} = t^{(\nu)} - \gamma^{\nu}(t) \equiv 0$ for every tensor field t of type (p,0) on M and $\nu < r$.

Remark 3.9. Let t be a tensor field of type (1,1) on M. According to Proposition 3.4 we have

$$(3.5) \qquad \begin{cases} t^{H,r}(X^H) = (tX)^H \\ t^{H,r}(X^{(\mu)}) = (tX)^{(\mu)} \end{cases}$$

$$(3.6) \qquad \begin{cases} t^{H,\nu}(X^H) = 0 \\ t^{H,\nu}(X^{(\mu)}) = (tX)^{(\nu+\mu-r)} \end{cases}$$

for $\nu = 1,\dots,r-1$, $\mu = 0,\dots,r-1$ and all vector fields X on M. Formulas (3.5) mean mean that $t^{H,r}$ coincides with the horizontal lift t^H of tensor fields of type (1,1) introduced in [5].

Horizontal r-lifts of geometric structures defined by tensor fields of type (1,1) were studied in [5].

4. Horizontal lifts of metrics

In this section we will study horizontal ν-lifts of tensor fields of type (0,2), particulary, horizontal ν-lifts of metrics and pseudometrics. At first, we introduce the following notation. Let g be a tensor field of type (0,p) on M and a = 1,...,p. For a vector field X on M we denote by $\alpha_X^a g$ the tensor field of type (0,p-1) on M given by the formula

$$(\alpha^a_X g)(X_1,\dots,X_{p-1}) = g(X_1,\dots,X_{a-1},X,X_a,\dots,X_{p-1})$$

for all vector fields $X_1,\dots,X_{p-1}$ on M.

From Proposition 3.2, Definition 3.5 and formula (1.13) we obtain immediately

Proposition 4.1. If g is a tensor field of type (0,2) on M and $\nu = 1,\dots,r$, then $g^{H,\nu}$ is given by the formulas

$$g^{H,\nu}(X^H,Y^H) = 0 \tag{4.1}$$

$$g^{H,\nu}(X^H,Y^{(\mu)}) = (\alpha^2_Y g)^{(\nu+\mu-r)}(X^H) \tag{4.2}$$

$$g^{H,\nu}(X^{(\mu)},Y^H) = (\alpha^1_X g)^{(\nu+\mu-r)}(Y^H) \tag{4.3}$$

$$g^{H,\nu}(X^{(\mu)},Y^{(\mu')}) = (g(X,Y))^{(\nu+\mu+\mu'-2r)} \tag{4.4}$$

for all vector fields X, Y on M and $\mu, \mu' = 0,\dots,r-1$.

Formulas (4.2) and (4.3) imply that if g is symmetric then so is $g^{H,\nu}$ for every $\nu = 1,\dots,r$.

Let g be a symmetric tensor field of type (0,2) (a quadratic form) on M. We suppose that the dimension of kernel of the linear mapping

$$T_xM \ni v \longrightarrow g(v,-) \in T^*_xM$$

is constant on M. We denote by c this dimension.

Every point of M has a neighborhood U and a frame $X_1,\dots,X_n$ defined on U such that

$$g(X_i,X_j) = 0 \qquad \text{for } i \neq j$$

$$g(X_i,X_i) = \begin{cases} 1 & \text{if } i = 1,\dots,a \\ -1 & \text{if } i = a+1,\dots,a+b \\ 0 & \text{if } i = a+b+1,\dots,n \end{cases}$$

for some numbers a, b; of course a + b + c = n = dim M.

We denote by (a,b,c) the signature of g. We suppose always that the numbers a and b are independent of a point of M. The frame $X_1,\dots,X_n$ is called adapted to g. We have the following proposition.

Proposition 4.2. If g is a symmetric tensor field of type (0,2) and (a,b,c) is the signature of g, then the signature of $g^{H,\nu}$ is

$$\left(\frac{a\nu + b(\nu+2)}{2}, \frac{a(\nu+2) + b\nu}{2}, c(\nu+1) + (r-\nu)m\right)$$

if ν is even, and

$$\left(\frac{(a+b)(\nu+1)}{2}, \frac{(a+b)(\nu+1)}{2}, c(\nu+1) + (r-\nu)n\right)$$

if ν is odd.

Proof. Let $X_1,\dots,X_n$ be an adapted frame to g on some neighborhood U. We denote by $g_0 = \left[g(X_i,X_j)\right]_{i,j=1,\dots,n}$

$$(4.6)\qquad g_0 = \begin{array}{c} \left[\begin{array}{ccccccccc}
1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & & 0 \\
\vdots & \ddots & & & & \vdots & & & \vdots \\
0 & & 1 & & & \vdots & & & 0 \\
0 & & & -1 & & 0 & & & 0 \\
\vdots & & & & \ddots & \vdots & & & \vdots \\
0 & & 0 & 0 & & -1 & & & 0 \\
0 & & 0 & 0 & & 0 & 0 & & 0 \\
\vdots & & & & & \vdots & & \ddots & \vdots \\
0 & & 0 & 0 & & 0 & 0 & & 0
\end{array}\right]
\begin{array}{l} \left.\begin{array}{c}\\ \\ \\ \end{array}\right\}a \\ \left.\begin{array}{c}\\ \\ \\ \end{array}\right\}b \\ \left.\begin{array}{c}\\ \\ \\ \end{array}\right\}c \end{array} \\
\underbrace{\hphantom{0\ \cdots\ 0}}_{a}\ \underbrace{\hphantom{0\ \cdots\ 0}}_{b}\ \underbrace{\hphantom{0\ \cdots\ 0}}_{c}
\end{array}$$

Now

$$(4.7)\qquad \left\{ X_i^H, X_j^{(\mu)} : i,j=1,\dots,n;\ \mu=0,\dots,r-1 \right\}$$

is a frame on $\pi^{-1}(U) \subset T^rM$. Using formulas (4.1) - (4.4) and Lemma 3.3 we find the matrix $\widetilde{G}$ of $g^{H,\nu}$ with respect to the frame (4.7)

$$\widetilde{G} = \begin{array}{c} \left[\begin{array}{cccccccc}
0 & \cdots & 0 & g_0 & A_1 & A_2 & \cdots & A_{\nu-1} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
g_0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
A_1^* & \cdots & 0 & 0 & 0 & 0 & \cdots & g_0 \\
A_2^* & \cdots & 0 & 0 & 0 & \cdots & g_0 & 0 \\
\vdots & & \vdots & & \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\
A_{\nu-1}^* & \cdots & 0 & 0 & g_0 & \cdots & \cdots & 0
\end{array}\right]
\begin{array}{l} \left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\}r-\nu+1 \\ \\ \left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\}\nu-1 \end{array} \\
\underbrace{\hphantom{A_{\nu-1}^*\ \cdots\ 0}}_{r-\nu+1}\quad \underbrace{\hphantom{g_0\ \cdots\ \cdots\ 0}}_{\nu-1}
\end{array}$$

where g_0 is given by (4.6) and${}_1 A_1,\dots,A_{\nu-1}$ are some $(n \times n)$-matrices (for a matrix A, A^* denotes the transpose of a matrix A). We also use the fact that for a constant function $f = g(X_i,X_j)$, $f^{(\lambda)} = 0$ if $\lambda \neq 0$ and $f^{(0)}$ is the same constant.

In order to calculate the signature of $\widetilde{G}$ we observe that $\widetilde{G} = P^*GP$, where

$$P = \left[\begin{array}{ccccccc} I & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & I & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & I & g_0 A_1 & \cdots & g A_{\nu-1} \\ 0 & \cdots & 0 & 0 & I & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & I \end{array}\right] \begin{array}{l} \\ r-\nu+1 \\ \\ \\ \\ \nu-1 \\ \\ \end{array}$$

$$\qquad \underbrace{\qquad\qquad}_{r-\nu+1} \qquad \underbrace{\qquad\qquad}_{\nu-1}$$

$$G = \left[\begin{array}{ccccccc} 0 & \cdots & 0 & g_0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ g_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & g_0 \\ \vdots & & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & 0 & 0 & g_0 & \cdots & 0 \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} R-\nu+1 \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \nu-1 \end{array}$$

$$\qquad \underbrace{\qquad\qquad}_{r-\nu+1} \qquad \underbrace{\qquad\qquad}_{\nu-1}$$

Since P is non-singular matrix and $\tilde{G} = P^{*}GP$, we conclude that $\tilde{G}$ and G have the same signature. To find the signature of G we look for the number of negative and positive solutions of tne equation

$$\det(G - \lambda I_0) = 0$$

where I_0 is the identity $(r + 1)n$-matrix.

Through a straighforward computation we can obtain that

$$\det(G-\lambda I_0) = \begin{cases} \pm(1-\lambda)^{\frac{a(\nu+2) + b\nu}{2}} (1+\lambda)^{\frac{a\nu + b(\nu+2)}{2}} \lambda^{c(\nu+1) + (r-\nu)n} & \text{if } \nu \text{ is even} \\ \pm(1-\lambda)^{\frac{(a+b)(\nu+1)}{2}} (1+\lambda)^{\frac{(a+b)(\nu+1)}{2}} \lambda^{c(\nu+1) + (r-\nu)n} & \text{if } \nu \text{ is odd} \end{cases}$$

which prove the proposition.

Proposition 4.2 implies that for $\nu < r$ the tensor $g^{H,\nu}$ is degenerate. As an immediate consequence of Proposition 4.2 we obtain

Corollary 4.3. If g is a pseudometric on M with signature $(a,b,0)$, then $g^{H,r}$ is a pseudometric on T^rM with signature

$$\left(\frac{ar + b(r+2)}{2} , \frac{a(r+1) + br}{2} , 0 \right)$$

if r is even, and

$$\left(\frac{(a+b)(r+1)}{2} , \frac{(a+b)(r+1)}{2} , 0 \right)$$

if r is odd.

We observe that $g^{H,r}$ is never positive-defined. In the case $r = 1$ Corollary 4.3 coincides with the result obtained by K. Yano and S. Ishihara [13].

5. A horozontal lift of linear connections

First of all, we prove the following theorem.

Theorem 5.1. If Γ is a connection of order r on M and ∇ is a linear connection on M, then there is one and only one linear connection ∇^H on T^rM such that

$$\nabla^H_{X^H} Y^H = (\nabla_X Y)^H \tag{5.1}$$

$$\nabla^H_{X^H} Y^{(\nu)} = [X^H, Y^{(\nu)}] \tag{5.2}$$

$$\nabla^H_{X^{(\nu)}} Y^H = 0 \tag{5.3}$$

$$\nabla^H_{X^{(\nu)}} Y^{(\mu)} = (\nabla_X Y)^{(\nu+\mu-r)} \tag{5.4}$$

for all vector fields X, Y on M and $\nu, \mu = 0,\dots,r-1$.

The linear connection ∇^H is called the horizontal lift of ∇ from M to T^rM with respect to Γ.

Proof. At first, we observe that conditions (5.1) - (5.4) determine uniquely ∇^H. Really, if ∇^H is a linear connection on T^rM which satisfies conditions (5.1) - (5.4), then we can compute the Christofell symbols of ∇^H as some functions of Christofell symbols of ∇ and Γ. This implies the unicity of ∇^H.

To prove the existence of such linear connection ∇^H on T^rM we consider a chart (U,x^i) on M. Let

$$\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$$

be the canonical frame on U. Now

$$\left\{ \left(\frac{\partial}{\partial x^i}\right)^H, \left(\frac{\partial}{\partial x^j}\right)^{(\nu)} \;:\; i,j=1,\dots,n;\ \nu=0,\dots,r-1 \right\}$$

is a frame on $T^rM|U$. Thus, there is one and only one linear connection $\tilde{\nabla}$ on T^rM U such that

$$\tilde{\nabla}_{(\frac{\partial}{\partial x^i})^H} (\frac{\partial}{\partial x^j})^H = (\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})^H \tag{5.5}$$

$$\tilde{\nabla}_{(\frac{\partial}{\partial x^i})^H} (\frac{\partial}{\partial x^j})^{(\nu)} = \left[(\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^{(\nu)} \right] \tag{5.6}$$

$$\tilde{\nabla}_{(\frac{\partial}{\partial x^i})^{(\nu)}} (\frac{\partial}{\partial x^j})^H = 0 \tag{5.7}$$

$$\tilde{\nabla}_{(\frac{}{\partial x^i})^{(\nu)}} (\frac{\partial}{\partial x^j})^{(\mu)} = (\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})^{(\nu+\mu-r)} \tag{5.8}$$

for $i,j = 1,\ldots,n$ and $\nu,\mu = 0,\ldots,r-1$.

Let K be the family of all pairs (X,Y) of vector fields on U such that conditions (5.1) - (5.4) hold for X, Y and $\nu,\mu = 0,\ldots,r-1$, where $\tilde{\nabla}^H$ is remplaced by $\tilde{\nabla}$. The definition of $\tilde{\nabla}$ implies that the pair $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ belongs to K for every $i,j = 1,\ldots,n$.

Now, we will prove that the family K has the following properties:

(a) If (X,Y) and (X',Y) belong to K, then so is (X+X',Y).

(b) If (X,Y) and (X,Y') belong to K, then so is (X,Y+Y').

(c) If (X,Y) belongs to K, then for every fonction f on M (fX,Y) and (X,fY) belong to K.

The properties (a) and (b) are an immediate consequence of the linearity of all operations which intervene in formulas (5.1) - (5.4).

To prove the property (c) we observe that $(fX)^H = f^{(0)} X^H$ and $X^H(f^{(0)}) = (Xf)^{(0)}$. Now, using these formulas and the fact that (X,Y) belongs to K, we obtain

$$\begin{aligned}
\tilde{\nabla}_{(fX)^H} Y^H &= \tilde{\nabla}_{f^{(0)} X^H} Y^H \\
&= f^{(0)} ({}_X Y)^H \\
&= (f \nabla_X Y)^H \\
\tilde{\nabla}_{X^H} (fY)^H &= \tilde{\nabla}_{X^H}(f^{(0)} Y^H) \\
&= X^H(f^{(0)}) + f^{(0)} \tilde{\nabla}_{Y^H} Y^H
\end{aligned}$$

$$= (Xf)^{(0)}\, Y^H + f^{(0)}\, (\nabla_X Y)^H$$
$$= (\nabla_X\, fY)^H$$

It means that the pairs (fX,Y) and (X,fY) verify condition (5.1). Now we have

$$\begin{aligned}
\tilde{\nabla}_{(fX)^H}\, Y^{(\nu)} &= f^{(0)}\, \tilde{\nabla}_{X^H}\, Y^{(\nu)} \\
&= f^{(0)}\, [X^H, Y^{(\nu)}] \\
&= [f^{(0)} X^H, Y^{(\nu)}] + Y^{(\nu)}(f^{(0)})\, X^H \\
&= [(fX)^H, Y^{(\nu)}]
\end{aligned}$$

because $Y^{(\nu)}(f^{(0)}) = (Yf)^{(\nu-r)} = 0$ for $\nu = 0,\dots,r-1$. According to (1.8), we obtain also

$$\begin{aligned}
\tilde{\nabla}_{X^H}\, (fY)^{(\nu)} &= \sum_{\mu=0}^{\nu} \tilde{\nabla}_{X^H}\, f^{(\mu)}\, Y^{(\nu-\mu)} \\
&= \sum_{\mu=0}^{\nu} X^H(f^{(\mu)})Y^{(\nu-\mu)} + f^{(\mu)}\, \tilde{\nabla}_{X^H}\, Y^{(\nu-\mu)} \\
&= \sum_{\mu=0}^{\nu} X^H(f^{(\mu)})Y^{(\nu-\mu)} + f^{(\mu)} [X^H, Y^{(\nu-\mu)} \\
&= \sum_{\mu=0}^{\nu} [X^H, f^{(\mu)} Y^{(\nu-\mu)}] \\
&= [X^H, (fY)^{(\nu)}]
\end{aligned}$$

Hence, condition (5.2) is verified by the pairs (fX,Y) and (X,fY) for $\nu = 0,\dots,r-1$.

A verification of condition (5.3) for the pairs (fX,Y) and (X,fY) is trivial. Condition (5.4) for these pairs we verify by the same method as in the construction of the complete lift $\nabla^{(r)}$ of ∇ (see A. Morimoto [8], [11]).

To sum up the last arguments we have proved properties (a), (b) and (c) of K. It implies that $\tilde{\nabla}$ is a linear connection on $T^rM|U$ satisfying conditions (5.1) - (5.4) for all vector fields X, Y on U and $\nu, \mu = 0,\dots,r-1$.

Let (U,x^i) and $(U',x^{i'})$ be two charts on M. Using our construction for these charts we obtain linear connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ on $T^rM|U$ and

$T^rM|U'$ respectively satifying conditions (5.1) - (5.4). It implies that the restrictions of $\tilde{\nabla}$ and $\tilde{\nabla}'$ to $T^rM|U \cap U'$ are two linear connectios on $T^rM|U \cap U'$ satisfying (5.1) - (5.4) for all vector fields X, Y on $U \cap U'$ and $\nu, \mu = 0, \dots, r-1$. Taking into account the unicity of linear connections satisfying these conditions, $\tilde{\nabla} = \tilde{\nabla}'$ on $T^rM|U \cap U'$. Thus, using an atlas on M we can construct a linear connection ∇^H on T^rM satisfying the conditions of our theorem.

From Theorem 5.1 we obtain

Corollary 5.2. If ∇ and ∇_0 are two linear connections on M, then there is one and only one linear connection ∇^H on TM (the horizontal lift of ∇ with respect to ∇_0) such that

$$\nabla^H_{X^H} Y^H = (\nabla_X Y)^H \quad , \quad \nabla^H_{X^H} Y^V = (\nabla_0{}_X Y)^V$$

$$\nabla^H_{X^V} Y^H = 0 \quad , \quad \nabla^H_{X^V} Y^V = 0$$

for all vector fields X, Y on M, where X^H denotes the horizontal lift of X with respect to ∇_0.

Proof. We employ Theorem 5.1 for r = 1. Taking into account that $X^{(0)} = X^V$ and $[X^H, Y^V] = (\nabla_{0X} Y)^V$, conditions (5.1) - (5.3) imply the first three conditions of our corollary and from (5.4) we obtain

$$\nabla^H_{X^V} Y^V = (\nabla_0{}_X Y)^{(0+0-1)} = (\nabla_0{}_X Y)^{(-1)} = 0$$

Corollary 5.3. The horizontal lift of ∇ with respect to ∇ to the tangent bundle TM coincides with the construction of K. Yano and S. Ishihara given in [12], [13].

Proof. We use Corollary 5.2 for $\nabla = \nabla_0$.

Proposition 5.4. Let ∇ be a linear connection and Γ be a connection of order r on M. If $\tilde{V}$ is a vertical vector field on T^rM and X is a vector field on M, then

$$\nabla^H_{\tilde{V}} X^H = 0 \quad , \quad \nabla^H_{X^H} \tilde{V} = [X^H, \tilde{V}]$$

Proof. To prove the first formula we fix a point y of T^rM. There are a number $\nu < r$ and a vector field Y on M such that

$$\tilde{V}(y) = Y^{(\nu)}(y)$$

Now we have

$$(\nabla^H_{\tilde{V}} X^H)(y) = (\nabla^H_{Y^{(\nu)}} X^H)(y) = 0$$

To prove the second formula we observe that every vertical vector field $\tilde{V}$ on T^rM can be locally written as a linear combination

$$\tilde{V} = \sum_a f_a Y_a^{(\nu_a)}$$

where $f^{(a)}$ are functions on T^rM, Y_a are vector fields on M and ν_a are numbers such that $0 \leqslant \nu_a \leqslant r-1$. Now from (5.1) we obtain

$$\begin{aligned}
\nabla^H_{X^H} \tilde{V} &= \sum_a \nabla^H_{X^H} f_a Y_a^{(\nu_a)} \\
&= \sum_a \left\{ X^H(f_a) Y_a^{(\nu_a)} + f_a \nabla^H_{X^H} Y^{(\nu_a)} \right\} \\
&= \sum_a \left\{ X^H(f_a) Y_a^{(\nu_a)} + f_a \left[X^H, Y_a^{(\nu_a)}\right] \right\} \\
&= \sum_a \left[X^H, f_a Y_a^{(\nu_a)}\right] \\
&= \left[X^H, \tilde{V}\right]
\end{aligned}$$

Proposition 5.5. Let T and $\tilde{T}$ be the torsions of ∇ and ∇^H respectively. If X, Y are vector fields on M and $\nu, \mu = 0, \ldots, r-1$, then

$$\tilde{T}(X^H, Y^H) = (T(X,Y))^H - R^{\Box}(X,Y) \tag{5.9}$$

$$\tilde{T}(X^H, Y^{(\nu)}) = 0 \tag{5.10}$$

$$\tilde{T}(X^{(\nu)}, Y^{(\mu)}) = (T(X,Y))^{(\nu+\mu-r)} \tag{5.11}$$

where $R^{\Box}(X,Y)$ is a vector field defined in [5] which depends on the curvature form of the given connection Γ of order r on M.

Proof. It is trivial taking into account the formula (see [5])

$$[X^H, Y^H] = [X,Y]^H + R^{\Box}(X,Y) \tag{5.12}$$

and the definition of ∇^H.

Proposition 5.6. Let Γ be a connection of order r, $r \geqslant 2$, and ∇ be a linear connection on M. The horizontal lift ∇^H of ∇ with respect to Γ is without torsion if and only if ∇ is without torsion and the curvature form Ω of Γ vanishes (Γ is without curvature).

Proof. In fact, if ∇^H is without torsion and $r \geqslant 2$, then there exist $\nu,\mu \leqslant r-1$ such that $\lambda = \nu+\mu-r > 0$. Now, from (5.11), we obtain $(T(X,Y))^{(\lambda)} = 0$ for some positive number λ and so $T = 0$. Next, from (5.9), we obtain $R^{\square}(X,Y) = 0$ for any vector fields X and Y on M. According to the definition of $R^{\square}(X,Y)$ (see [5]), we have $\Omega(X^H,Y^H) = 0$, where Ω is the curvare form of Γ and X^H and Y^H denote the horizontal lifts (with respect to Γ) of X and Y to the bundle F^rM of frames of order r. From this we obtain $\Omega = 0$.

Inversely, if we suppose that $T = 0$ and $\Omega = 0$, then the definition of $R^{\square}(X,Y)$ implies that $R^{\square}(X,Y) = 0$ and from (5.9) - (5.12) we obtain $\tilde{T} = 0$. The prof is finished.

In the case $r = 1$ this proposition is not true because in formula (5.11) we must consider $\nu = \mu = 0$ and we have

$$\tilde{T}(X^{(0)},Y^{(0)}) = (T(X,Y))^{(-1)} = 0 \tag{5.13}$$

In the case $r = 1$, using (5.9), (5.10) and (5.13) we can prove easily the following proposition.

Proposition 5.7. Let ∇, ∇_0 be two linear connections on M. We denote by T and R_0, respectively, the torsion of ∇ and the curvature of ∇_0.

(a) If $T = 0$, then the horizontal lift ∇^H of ∇ with respect to ∇_0 is without torsion if and only if $R_0 = 0$.

(b) If $R_0 = 0$, then ∇^H is without torsion if and only if ∇ is without torsion.

In the case $\nabla = \nabla_0$ the part (a) of Proposition 5.7 was proved by K. Yano and S. Ishihara [12].

We can look for the curvature tensor of ∇^H but formulas are more complicated. Namely, we have

Proposition 5.8Let ∇ be a linear connection and Γ be a connection of order r on M. If $\tilde{R}$ is the curvature tensor of the horizontal lift ∇^H of ∇ with respect to Γ, then

$$\tilde{R}(X^H,Y^H)Z^H = (R(X,Y)Z)^H$$

$$\tilde{R}(X^H,Y^H)Z^{(\nu)} = \left[R^{\square}(X,Y),Z^{(\nu)}\right] - \nabla^H_{R^{\square}(X,Y)} Z^{(\nu)}$$

$$\tilde{R}(X^H,Y^{(\nu)})Z^H = 0$$

$$\tilde{R}(X^H,Y^{(\nu)})Z^{(\mu)} = \left[X^H,[Y,Z]^{(\nu+\mu-r)}\right] - \nabla^H_{Y^{(\nu)}}\left[X^H,Z^{(\mu)}\right] - \nabla^H_{[X^H,Y^{(\nu)}]}Z^{(\mu)}$$

$$R(X^{(\nu)},Y^{(\mu)})Z^H = 0$$

$$R(X^{(\nu)},Y^{(\mu)})Z^{(\eta)} = (R(X,Y)Z))^{(\nu+\mu+\eta-2r)}$$

for all vector fields X, Y, Z on M and $\nu,\mu,\eta = 0,\dots,r-1$.

Proof. Trivial from (5.1) - (5.4) and the definitions, taking into account that $[X^H,Y^{(\nu)}] = 0$ for $\nu \leq r-1$.

6. Relationship between the horizontal lifts of linear connection and tensor fields.

Let Γ be a given connection of order r on M. This connection Γ defines the covariant derivation of sections of natural bundles of order r. We will denote this derivation by $D^{(r)}$. For every $\nu = 0,\dots,r-1$, this connection Γ determines one and only one connection of order ν, called the part of order ν of Γ, and this connection defines the covariant derivation, denoted by $D^{(\nu)}$, of sections of natural bundles of order ν.

We consider the natural (vector) bundle

$$J^{\lambda}(TM) = \{j_x^{\lambda}X : x \in M, \text{ X is a vector field on M}\}$$

of order $\lambda+1$, where $\lambda = 0,\dots,r-1$. If σ is a section of $J^{\lambda}(TM)$ and $\nu \leq \lambda$ we can define the vector field $\sigma^{(\nu)}$ on T^rM by

$$(6.1) \qquad \sigma^{(\nu)}(y) = X^{(\nu)}(y)$$

where y is a point of T^rM and X is a vector field on M such that

$$\sigma(\pi(y)) = j^{\lambda}_{\pi(y)}X \quad .$$

Of course, we have $(J^{\nu}X)^{(\nu)} = (J^{r-1}X)^{(\nu)} = X^{(\nu)}$ for $\nu = 0,\dots,r-1$, where $J^{\nu}X$ is the section $x \longrightarrow j^{\nu}_xX$ of $J^{\nu}(TM)$.

In [5] the following formula was proved

$$(6.2) \qquad [X^H,Y^{(\nu)}] = (D_X^{(\nu+1)}J^{\nu}X)^{(\nu)} = (D_X^{(r)}J^{r-1}Y)^{(\nu)}$$

for $\nu = 0,\dots,r-1$. The last equality holds because in the expression $D_X^{(\lambda+1)}J^{\lambda}Y$ we can use any number $\lambda \geq \nu$.

We also consider the (vector) bundle

$$J^{\lambda}(TM\otimes T^*M) = \{j_x^{\lambda}t : x \in M, \text{ t is a tensor field of type (1,1) on M}\}$$

of order $\lambda+1$ for $\lambda = 0,\dots,r-1$. We can define an operation between sections of $J^\lambda(TM)$ and $J^\lambda(TM \otimes T^*M)$. If σ is a section of $J^\lambda(TM)$ and τ is a section of $J^\lambda(TM \otimes T^*M)$, we consider a new section $\tau\sigma$ of $J^\lambda(TM)$ given by

$$(6.3) \qquad (\tau\sigma)(x) = j^\lambda_x(t(X))$$

where x is a point of M, t is a tensor field of type $(1,1)$ on M such that $\tau(x) = j^\lambda_x t$ and X is a vector field on M such that $\sigma(x) = j^\lambda_x X$.

Taking into account the bilinearity of the operation

$$(\tau, \sigma) \longrightarrow \tau\sigma$$

we have the following formula

$$(6.4) \qquad D_X^{(\lambda+1)}(\tau\sigma) = (D_X^{(\lambda+1)}\tau)\sigma + \tau(D_X^{(\lambda+1)}\sigma) \ .$$

If τ is a section of $J^\lambda(TM \otimes T^*M)$ and $\nu \leq \lambda$, then we define the tensor field $\tau^{(\nu)}$ of type $(1,1)$ on T^rM setting

$$(6.5) \qquad \tau^{(\nu)}(y) = t^{(\nu)}(y)$$

where y is a point of T^rM and t is a tensor field of type $(1,1)$ on M such that $\tau(\pi(y)) = j^\lambda_{\pi(y)} t$.

It is clear that for any tensor field t of type $(1,1)$ on M we have

$$(6.6) \qquad t^{(\nu)} = (J^\nu t)^{(\nu)} = (J^{r-1} t)^{(\nu)}$$

for $\nu = 0,\dots,r-1$, where $J^\nu t$ is the section $x \longrightarrow j^\nu_x t$ of $J^\nu(TM \otimes T^*M)$..

We have the following proposition.

Proposition 6.1. Let α be integer from 1 to r. If If t is a tensor field of type $(1,1)$ on M, X, Y are vector fields on M and $\nu, \mu = 0,\dots,r-1$, then we have

$$(6.7) \qquad (\nabla^H_{X^H} t^{H,\alpha})(X^H) = \begin{cases} 0 & \text{if } \alpha \leq r-1 \\ ((\nabla_X t)(Y))^H & \text{if } \alpha = r \end{cases}$$

$$(6.8) \qquad (\nabla^H_{X^H} t^{H,\alpha})(Y^{(\nu)}) = (D_X^{(r)} J^{r-1} t)^{(\alpha)}(Y^{(\nu)})$$

$$(6.9) \qquad (\nabla^H_{X^{(\nu)}} t^{H,\alpha})(Y^H) = 0$$

$$(6.10)\qquad (\nabla^H_{X^{(\nu)}} t^{H,\alpha})(Y^{(\mu)}) = (\nabla_X t)^{(\alpha+\nu-r)}(Y^{(\mu)})$$
$$= ((\nabla_X t)(Y))^{(\alpha+\nu+\mu-2r)}$$

To prove this proposition we need the following lemma.

Lemma 6.2. If σ is a section of $J^{r-1}(TM)$ and $\nu \leqslant r-1$, then

$$t^{H,\alpha}(\sigma^{(\nu)}) = (t\sigma)^{(\alpha+\nu-r)}$$

where $t\sigma = (J\, t)\sigma$ (see (6.3)).

Proof. Let y be a point of T^rM and X be a vector field on M such that $\sigma(\pi(y)) = j^{r-1}_{\pi(y)}X$. Now $\sigma^{(\nu)}(y) = X^{(\nu)}(y)$ and according to (3.5) and (3.6) we obtain

$$(t^{H,\alpha}(\sigma^{(\nu)}))(y) = (t^{H,\alpha}(X^{(\nu)}))(y)$$
$$= (tX)^{(\alpha+\nu-r)}$$
$$= (t\sigma)^{(\alpha+\nu-r)}$$

Proof of Proposition 6.1. Formula (6.7) is an immediate consequence of (3.5) and (3.6). To prove (6.8) we use (3.5), (3.6), (5.2), (6.2) and Lemma 6.2. Really, we obtain

$$(\nabla^H_{X^H} t^{H,\alpha})(Y^{(\nu)}) = \nabla^H_{X^H}(t^{H,\alpha}(Y^{(\nu)})) - t^{H,\alpha}(\nabla^H_{X^H}\, Y^{(\nu)})$$
$$= \nabla^H_{X^H}(tY)^{(\alpha+\nu-r)} - t^{H,\alpha}((D^{(r)}_X J^{r-1}Y))^{(\nu)}$$
$$= D^{(r)}_X(J^{r-1}(tY))^{(\alpha+\nu-r)} - (t(D^{(r)}_X J^{r-1}Y))^{(\alpha+\nu-r)}$$
$$= (D^{(r)}_X J^{r-1}t)^{(\alpha)}(Y^{(\nu)}).$$

Formula (6.9) is obtained directly from definitions of ∇^H and $t^{H,\alpha}$. We verify (6.19) in the same way as in the paper of A. Morimoto [11]. The proof is complete.

From Proposition 6.1 we deduce immediately

Corollary 6.3. Let Γ be a connection of order r, ∇ be a linear connection and t be a tensor field of type (1.1) on M. $t^{H,\alpha}$ is parallel with respect to ∇^H if and only if t is parallel with respect to ∇ and $D^{(r)}J^{r-1}t = 0$, where $D^{(r)}$ is the covariavt derivation with respect to Γ.

Corollary 6.4. (K. Yano and S. Ishihara [12]) Let ∇ be a linear connection and t a tensor field of type (1,1) on M. If ∇^H and $t^H = t^{H,1}$ denote the horizontal lifts of ∇ and t to the tangent bundle TM with respect to ∇ , then $\nabla^H t^H = 0$ if and only if $\nabla t = 0$.

We have also the following proposition.

Proposition 6.5. If t is a tensor field of type (1,1) on M, X, Y are vector fields on M and $\nu, \mu = 0, \dots, r-1$, then

$$(L_{X^H} t^{H,\alpha})(Y^H) = \begin{cases} - t^{(\alpha)}(R^{\square}(X,Y)) & \text{if } \alpha \leq r-1 \\ ((L_X t)(Y))^H + R^{\square}(X,Y) - t^{(r)}(R^{\square}(X,Y)) & \text{if } \alpha = r \end{cases}$$

$$(L_{X^H} t^{H,\alpha})(Y^{(\nu)}) = (D_X^{(r)} J^{r-1} t)^{(\alpha)}(Y^{(\nu)})$$

$$(L_{X^{(\nu)}} t^{H,\alpha})(Y^H) = \begin{cases} (t\ D_Y^{(r)} J^{r-1} X)^{(\alpha+\nu-r)} & \text{if } \alpha \leq r-1 \\ (t\ D_Y^{(r)} J^{r-1} X - D_{tY}^{(r)} X)^{(\nu)} & \text{if } \alpha = r \end{cases}$$

$$(L_{X^{(\nu)}} t^{H,\alpha})(Y^{(\mu)}) = ((L_X t)(Y))^{(\alpha+\nu+\mu-2r)}$$

Proof. Using the definitions of $t^{H,\alpha}$ and ∇^H and formula (6.2) we obtain directly the above formulas taking into account that $t^{H,\alpha}$ and $t^{(\alpha)}$ coincide for vertical vectors. The last formula can be obtained as in the paper of A. Morimoto [7], [11].

From Proposition 6.5 we obtain immrdiately

Corollary 6.6. Let r be an integer such that $r \geqslant 2$.

(a) If $\alpha \leqslant r-1$, then $L_{X^{(\nu)}} t^{H,\alpha} = 0$ if and only if $L_X t = 0$ and for every vector field Y on M we heve $t(D_Y^{(r)} J^{r-1} X) = 0$.

(b) $L_{X^{(\nu)}} t^{H,r} = 0$ if and only if $L_X t = 0$ and for every vector field Y Y on M we have $t(D_Y^{(r)} J^{r-1} X) = D_{tY}^{(r)} J^{r-1} X$.

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J. GANCARZEWICZ, INSTYTUT MATEMATYKI, UNIWERSYTET JAGIELLOŃSKI,
ul. REYMONTA 4, 30-059 KRAKÓW, POLAND
M. SALGADO, DEPT. GEOMETRIA y TOPOLOGIA, FACULTAD de MATEMÁTICAS,
UNIVERSIDAD de SANTIAGO, SANTIAGO de COMPOSTELA,
SPAIN