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# CONFORMAL TRANSFORMATION, CONFORMAL CHANGE, AND CONFORMAL COVARIANTS

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Abstract. Though standard Riemannian $S^4$ and standard Lorentzian $S^1 \times S^3$ support natural, fourth-order differential operators on one-forms which intertwine representations of their 15-dimensional conformal groups, there is no general fourth-order differential operator on one-forms in Riemannian or pseudo-Riemannian manifolds that is universally covariant under conformal change of metric.

**0. Introduction.** Conformal transformation and conformal change of metric have long been important topics in Mathematics and Physics, and with the advent of anomaly and string theories, this importance can only increase. There is an immense literature on the classification problem for differential operators covariant under conformal *transformation*, in, for example, 4-dimensional Minkowski, de Sitter, and compactified Minkowski space. Such operators are *intertwining* for representations of the 15-dimensional conformal group, and thus have obvious importance for anyone seeking to decompose the natural tensor-spinor representations on these spaces (that is, to find elementary particles without expensive equipment).

At the same time, differential geometers have long been interested, albeit sometimes peripherally, in differential operators covariant under conformal *change of metric*, usually in the Riemannian (positive definite metric) case. The best-known example is the *Yamabe operator* $D_2 = \Delta + (n-2)K/4(n-1)$ on scalar functions ($\Delta$ = Laplace-Beltrami, $n$ = dimension, $K$ = scalar curvature), the conformal covariance law for which was used essentially and repeatedly in the solution of the *Yamabe problem* of conformal deformation to constant scalar curvature [**Y, T, A,**

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**Sc, LP**]. In dimension 4, $D_2$ is the analytic continuation in signature of the curvature-modified d'Alembertian $\Box + K/6$ of classical relativity; indeed, classical relativity *is* the 4-dimensional, Lorentzian Yamabe problem.

Covariants of conformal change are, of course, not differential operators on fixed spaces, but operator *schemes* which assign a differential operator to each Riemannian (or pseudo-Riemannian) manifold in a natural, universal manner. Given a covariant of conformal change, we get a covariant of conformal transformation on any manifold with metric, though the property is usually vacuous: generically, there are no conformal transformations but the identity. Nevertheless, there are important manifolds with large conformal groups; the maximal dimension $(n+1)(n+2)/2$ is attained for the product spaces $S^p \times S^q, p+q=n$, with the standard signature $(q,p)$ metric; in fact, the conformal group is isomorphic to $O(p,q)$.

There is no obvious reason, though, that covariants of conformal transformation on these very symmetric manifolds should always give rise to covariants of conformal change on general manifolds with metric. Indeed, the conformal changes actually implemented by conformal transformations cut out only a finite-dimensional submanifold $N$ in the infinite-dimensional manifold $C_+^\infty(M)$ of all conformal changes, and one suspects (correctly, as we show here) that $N$ is not "discriminating enough" to keep out all operators which do not generalize, even for very symmetric $M$ (large *dim* $N$).

In this paper, we give an example of an operator $D_{4,1}$ previously studied [**B2**, Sec. 3] in the settings of Riemannian $S^4$ and Lorentizian $S^1 \times S^3$, which, though covariant under conformal transformation, has no generalization to a covariant of conformal change on Riemannian or Lorentzian 4-manifolds. $D_{4,1}$ acts on 1-forms, has order 4, and fits into a large, many-parametered class of intertwining operators for $O(1,5)$ and $O(2,4)$ [**J**, **B2**] , and indeed for $O(2,p)$ [**B2**, **B3**]. It generalizes to a covariant of conformal change in dimensions $n \neq 1,2,4$ [**B1**] . From this last point of view, our result here is that the restriction $n \neq 4$ is essential: not only does the formula in [**B1**] not give a conformal covariant of the type desired, but there is no such covariant.

The special behavior of dimension 4 relative to this and other operators is intimately related to the (relative) conformal invariance of the *Bach tensor* [**Ba**] . In fact, the argument in [**B1**] *does* give an operator in dimension 4 of the right *level* (homogeneity under uniform dilation of the metric), but the *order* collapses to 0; the operator is just the pointwise action of the Bach 2-tensor on 1-forms. Alternatively, analytic continuation of the calculation of [B1] in $n$, followed by division by $n-4$, gives the Bach tensor and its conformal properties. In this paper, the Bach tensor also enters in an essential way: being separately covariant in its action on 1-forms, it is not available to help compensate the conformal non-covariance of higher-order terms in our prospective operator (Theorem 4.11).

Sections 1-3 are introductory; in Section 4, the main non-existence result is proved via a series of lemmas. Among other things, these lemmas show that modulo zeroth-order operators, the choices made in [B1, Sec. 2.1] in the construction of a fourth-order conformal covariant on forms are essential.

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**1. Covariants of conformal transformation and change.** Let $(M,g)$ be a connected pseudo-Riemannian manifold of dimension $n$. If $M$ is oriented, denote by $E$ a choice of normalized volume form, and if $M$ has spin structure, denote by $\gamma$ a choice of spin representation. A *conformal transformation* of $(M,g)$ is diffeomorphism $h$ of $M$ which has the effect

$$h\cdot g=\Omega_h^2 g \quad 0<\Omega\in C^\infty(M)$$

on the metric. ($h\cdot$ will always denote the natural push-forward of tensor fields by a diffeomorphism as in [H, p. 90]. For covariant tensors like $g$, $h\cdot$ acts as $(h^{-1})^*$. For conformal $h$, $h\cdot$ extends to tensor-spinors [Ko].) The conformal transformations form a Lie group $\mathcal{C}(M,g)$ of dimension at most $(n+1)(n+2)/2$ [KN, Notes 9, 11, 13], and for any vector bundle $F$ of tensor-spinors,

$$u_a(h): h\longmapsto \Omega_h^a h\cdot \tag{1.1}$$

is a homomorphism $\mathcal{C}(M,g)\longrightarrow Aut\ C^\infty(F)$. A differential operator $D$ is *covariant under conformal transformation* if for some $a,b\in\mathbf{R}$, $Du_a(h)=\pm u_b(h)D$ for all $h\in\mathcal{C}(M,g)$. The $\pm$ sign comes into play only if $D$ depends on our choice of $E$ and/or $\gamma$ (both determined up to a sign). Then, clearly, we get $Du_a(h)=u_b(h)D$ for $h$ in the identity component of $\mathcal{C}(M,g)$, with predictable sign changes for the other components, depending on the signs in $h\cdot E=\pm\Omega_h^n E$, $h\cdot\gamma=\pm\Omega_h^{-1}\gamma$.

On the other hand, if $D$ is a differential operator *scheme*, a universal assignment of a differential operator to each pseudo-Riemannian manifold of dimension $n$ which, in local coordinates, is built universally and polynomially from $g$, its partial derivatives, its inverse $g^{\#} = (g^{\alpha\beta})$, and if applicable, $E, \gamma$, we say that $D$ is *covariant under conformal change* if

$$\overline{g} = \Omega^2 g \quad (\overline{E} = \Omega^n E,\ \overline{\gamma} = \Omega^{-1}\gamma), \qquad 0 < \Omega \in C^\infty(M) \Rightarrow \overline{D} = \Omega^{-b} D \Omega^a \tag{1.2}$$

for some $a, b \in \mathbb{R}$. (Here $\Omega^a$ is to be interpreted as a multiplication operator.) This second kind of covariance is more general than the first: given $M, g$ (and possibly $E, \gamma$) and $h \in \mathcal{C}(M, g)$, consider the composition

$$(M, g) \xrightarrow{h} (M, \overline{g} = h \cdot g = \Omega_h^2 g) \xrightarrow{id} (M, g)$$

of an isometry and a conformal change. If $D$ is as in (1.2),

$$D\Omega_h^a h\cdot = \Omega_h^b \overline{D} h\cdot = \pm \Omega_h^b h \cdot D,$$

the isometry invariance or anti-invariance used in the second equality coming from the universality of $D$.

Because conformal changes live in one-parameter *groups*, and just not one-parameter *families*, (1.2) is equivalent to an infinitesimal property [**B1**, Proposition 1.12] : let everything depend in a jointly smooth way on $x \in M$ and $u$ in some open interval about 0 in $\mathbb{R}$, and let $^\cdot = (\partial/\partial u)|_{u=0}$. Then (1.2) is equivalent to

$$\dot{g} = 2\omega g,\ \omega \in C^\infty(M) \Rightarrow \dot{D} = -(b-a)\omega D + a[D, \omega]. \tag{1.3}$$

**2. The operator on $S^4$ and on $S^1 \times S^3$.** Among the conformal transformation covariants studied in [**B2**] is an operator $D_{4,1}$ on 1-forms in Riemannian $S^4$, and its Lorentz signature counterpart (also called $D_{4,1}$) on $S^1 \times S^3$. The leading term in each case is $3\delta d\delta d - d\delta d\delta$, $d$ being the exterior derivative, and $\delta$ its formal adjoint in the appropriate metric. $D_{4,1}$ has *conformal bidegree* $(-1, 3)$; that is, it intertwines the representations $u_{-1}$ and $u_3$ of (1.1). According to [**B2**, Remark 3.30],

$$D_{4,1} = 3\delta d(\delta d - 2) - d\delta(d\delta + 2) \quad \text{on } S^4.$$

(Refer to [**B2**, proof of Theorem 3.2] for the definitions of the operators $B$ and $C$ used in [**B2**, Remark 3.30].) On $S^1 \times S^3$, $D_{4,1}$ is best expressed in terms of the exterior derivative $d_3$ and coderivative $\delta_3$ on $S^3$, and partial differentiation $\partial_t$ with respect to the $S^1$ parameter $t$. If $\phi \in C^\infty(\Lambda^1(S^1 \times S^3))$, we can write $\phi = \phi_0 dt + \phi_1$, where $\phi_0 \in C^\infty(S^1 \times S^3)$, and $\phi_1 \in C^\infty(\Lambda^1(S^1 \times S^3))$ has $\iota(dt)\phi_1 = 0$ (interior multiplication). Unravelling [**B2**, equation (3.21)], we get

$$
\begin{aligned}
&(D_{4,1}\phi)_0 = \\
&\qquad (-\partial_t^4 + 2\delta_3 d_3 \partial_t^2 + 3\delta_3 d_3 \delta_3 d_3 - 10\partial_t^2 - 6\delta_3 d_3 - 9)\phi_0 \\
&\qquad\quad - 4(\partial_t^2 + \delta_3 d_3 + 1)\delta_3 \partial_t \phi_1, \\
&(D_{4,1}\phi)_1 = \\
&\qquad (3\partial_t^4 + (6\delta_3 d_3 + 2d_3\delta_3)\partial_t^2 + 3\delta_3 d_3 \delta_3 d_3 - d_3\delta_3 d_3 \delta_3 \\
&\qquad\quad + 6\partial_t^2 - 6\delta_3 d_3 + 2d_3\delta_3 + 3)\phi_1 \\
&\qquad\quad - 4(\partial_t^2 + d_3\delta_3 + 1)d_3 \partial_t \phi_0.
\end{aligned}
$$

By [**B2**, equation (3.2)], this has leading term $3\delta d\delta d - d\delta d\delta$.

**3. Tensor bundles, structure groups and their representations, and differential operators.** We would like to show that there is no covariant of conformal change on general 4-manifolds of any metric signature which acts on 1-forms and is *properly* fourth-order (never of lower order on any manifold). By analytic continuation in signature [**BØ**, Sec. 7], it is enough to work with Riemannian metrics. On a Riemannian 4-manifold, the structure group of any tensor bundle can be reduced from $GL(4)$ to $O(4)$, and in the orientable case to $SO(4)$. The irreducible representations of $O(4)$ (resp. $SO(4)$ )are parameterized by *highest weights* $(p, q) \in \mathbb{Z}^2$ with $p \geq q \geq 0$ (resp. $p \geq |q|$). Each irreducible representation corresponds to a vector bundle $V(p, q)$ which is irreducible under its structure group. We abbreviate $V(p, 0)$ by $V(p)$. For example, the trivial line bundle is $\Lambda^0 \cong V(0)$; the 1-forms $\Lambda^1 \cong V(1)$; and the trace-free symmetric $p$-tensors $TFS^p \cong V(p)$, where $\cong$ stands for $O(4)$- or $SO(4)$-equivariant bundle isomorphism. The trace-free tensors with the symmetries of the Riemann tensor (the algebraic Weyl tensors) live in

$$Weyl \cong_{O(4)} V(2,2),$$

and thus the full bundle of tensors with Riemann symmetry is

$$Curv \cong_{O(4)} V(0) \oplus V(2) \oplus V(2,2), \tag{3.1}$$

where the summands are represented by the scalar, trace-free Ricci, and Weyl curvatures.

Under $SO(4)$, the tensor product of $V(1)$ with $V(p, q)$ breaks up into irreducibles as follows:

(3.2)
$$\begin{aligned}
V(1)\otimes V(0) &\cong_{SO(4)} V(1),\\
V(1)\otimes V(p) &\cong_{SO(4)} V(p-1)\oplus V(p,-1)\oplus V(p,1)\oplus V(p+1),\\
&\qquad p\geq 1,\\
V(1)\otimes V(p,q) &\cong_{SO(4)} V(p-1,q)\oplus V(p,q-1)\oplus\\
&\qquad V(p,q+1)\oplus V(p+1,q),\ p>|q|\geq 1,\\
V(1)\otimes V(p,p) &\cong_{SO(4)} V(p,p-1)\oplus V(p+1,p),\ p\geq 1,\\
V(1)\otimes V(p,-p) &\cong_{SO(4)} V(p,-p+1)\oplus V(p+1,-p),\ p\geq 1.
\end{aligned}$$

(See, e.g., [**F**, Theorem 3.4].) If $q \geq 1$, the $O(4)$-bundle $V(p,q)$ is the direct sum of the $SO(4)$-bundles $V(p,q)$ and $V(p,-q)$; $V(p)$ is the same for $O(4)$ and $SO(4)$. (In particular, the 2-forms $\Lambda^2$ are $V(1,1)$ as an $O(4)$-bundle, and $V(1,1)\oplus V(1,-1)$ as an $SO(4)$-bundle.) Thus

$$\begin{aligned}
V(1)\otimes V(0) &\cong_{O(4)} V(1),\\
V(1)\otimes V(p) &\cong_{O(4)} V(p-1)\oplus V(p,1)\oplus V(p+1),\ p\geq 1,\\
V(1)\otimes V(p,q) &\cong_{O(4)} V(p-1,q)\oplus V(p,q-1)\oplus\\
&\qquad V(p,q+1)\oplus V(p+1,q),\ p>q>1,\\
V(1)\otimes V(p,1) &\cong_{O(4)} V(p-1,1)\oplus 2V(p)\oplus V(p+1,1),\ p>1,\\
V(1)\otimes V(1,1) &\cong_{O(4)} 2V(1)\oplus V(2,1),\\
V(1)\otimes V(p,p) &\cong_{O(4)} V(p,p-1)\oplus V(p+1,p),\ p>1.
\end{aligned}\tag{3.3}$$

(3.2) and (3.3) are especially important here for two reasons: first, we are concerned with differential operators on $V(1)$, and second, the covariant derivative $\nabla$ carries sections of $V(p,q)$ to sections of $V(1)\otimes V(p,q)$.

By (3.1) and (3.3), the covariant derivative of a section of $Curv$ is a section of $2V(1)\oplus 2V(2,1)\oplus V(3)\oplus V(3,2)$; but by the second Bianchi identity, the covariant derivative of the actual Riemann tensor lives in

$$\nabla Curv \cong_{O(4)} V(1)\oplus V(2,1)\oplus V(3)\oplus V(3,2). \tag{3.4}$$

(See, e.g., [**St**].) The $V(1)$ summand is represented by the gradient of the scalar curvature; the $V(3)$ component is the trace-free symmetric part of the covariant derivative $\nabla r$ of the Ricci tensor; $\nabla r$ and the covariant derivative $\nabla C$ of the Weyl tensor both contribute to the $V(2,1)$ piece, and the $V(3,2)$ piece is a projection of $\nabla C$.

It will be important to us to have explicit formulas for the projections of the covariant derivative of the trace-free Ricci tensor onto the summands of

$$\Lambda^1 \otimes TFS^2 \cong_{O(4)} V(1) \oplus V(2,1) \oplus V(3).$$

It is easy to get the $V(1)$ and $V(3)$ components,

$$\begin{aligned} Proj_{V(3)}(\varphi_{\alpha\beta\gamma}) &= \left(\tfrac{2}{9}(g_{\alpha\beta}\varphi^{\lambda}{}_{\lambda\gamma} + g_{\alpha\gamma}\varphi^{\lambda}{}_{\lambda\beta} - \tfrac{1}{2}g_{\beta\gamma}\varphi^{\lambda}{}_{\lambda\alpha})\right), \\ Proj_{V(3)}(\varphi_{\alpha\beta\gamma}) &= \left(\tfrac{1}{3}(\varphi_{\alpha\beta\gamma} + \varphi_{\beta\alpha\gamma} + \varphi_{\gamma\alpha\beta}) \right. \\ &\quad \left. - \tfrac{1}{9}(g_{\alpha\beta}\varphi^{\lambda}{}_{\lambda\gamma} + g_{\alpha\gamma}\varphi^{\lambda}{}_{\lambda\beta} + g_{\beta\gamma}\varphi^{\lambda}{}_{\lambda\alpha})\right), \end{aligned} \tag{3.5}$$

and thus the $V(2,1)$ component is

$$\begin{aligned} Proj_{V(2,1)}(\varphi_{\alpha\beta\gamma}) &= \left(\tfrac{1}{3}(2\varphi_{\alpha\beta\gamma} - \varphi_{\beta\alpha\gamma} - \varphi_{\gamma\alpha\beta}) \right. \\ &\quad \left. - \tfrac{1}{9}(g_{\alpha\beta}\varphi^{\lambda}{}_{\lambda\gamma} + g_{\alpha\gamma}\varphi^{\lambda}{}_{\lambda\beta} - 2g_{\beta\gamma}\varphi^{\lambda}{}_{\lambda\alpha})\right). \end{aligned} \tag{3.6}$$

By Weyl's invariant theory [**W, ABP, DP**] , all $O(4)$-differential operators are built polynomially from $g, g^{\#} = (g^{\alpha\beta})$, $\nabla$, the Riemann tensor $R$, contractions, permutation of arguments (indices), and tensor product. For $SO(4)$-operators, $E$ can also enter, for example, via the Hodge $*$. (To get the operator version of this characterization from the tensor quantity version, take the leading symbol, form an appropriate "leading term" with $\nabla$, subtract, and iterate.) Any properly fourth-order operator scheme $D$ on a tensor bundle has a leading term in 4 covariant derivatives; otherwise its order on Riemann flat manifolds would be less than 4. This implies that the *level* of $D$, if $D$ is to be conformally covariant, is 4:

$$\overline{g} = A^2 g,\ 0 < A \in \mathbb{R} \Rightarrow \overline{D} = A^{-4}D,$$

by the dilation laws for $\nabla$ and $R$. Thus all such $D$ are sums of terms which are schematically

$$\begin{array}{rl} \nabla\nabla\nabla\nabla & \text{(fourth order)}, \\ R\nabla\nabla & \text{(second order)}, \\ (\nabla R)\nabla & \text{(first order)}, \\ (\nabla\nabla R),\, RR & \text{(zeroth order)}. \end{array} \tag{3.7}$$

**4. A nonexistence result.** In this section, we always work over Riemannian 4-manifolds. By *a* $D_{4,1}$ we shall mean a properly fourth-order covariant of conformal change acting on 1-forms. This should *a priori* only be assumed to be $SO(4)$-equivariant, but we shall soon show

(Lemma 4.4) that if there is an $SO(4)$-operator of the type we seek, then there is also an $O(4)$-operator. We let $C$ denote the Weyl conformal curvature tensor, and

$$\begin{aligned} J &= K/6, \\ V_{\alpha\beta} &= \tfrac{1}{2}(r_{\alpha\beta} - Jg_{\alpha\beta}), \\ T_{\alpha\beta} &= V_{\alpha\beta} - \tfrac{1}{4}Jg_{\alpha\beta}. \end{aligned}$$

$T$ is the trace-free Ricci tensor, and $J$ and $V$ are useful for their conformal variational properties: in terms of the $^{\boldsymbol{\cdot}}$ operator of Sec. 1,

$$\begin{aligned} \dot{J} &= -2\omega J + \Delta\omega, \\ \dot{V}_{\alpha\beta} &= -\nabla_\alpha\nabla_\beta\omega = -(Hess\ \omega)_{\alpha\beta} \end{aligned} \tag{4.1}$$

[**B1**, Sec. 1.d] .

LEMMA 4.1. *Modulo second-order operators, the space of fourth-order $SO(4)$-differential operator schemes is 2-dimensional, generated by $\delta d\delta d$ and $d\delta d\delta$.*

PROOF: The fourth-order symbol of such an operator $D$ is a 4-homogeneous $SO(4)$-bundle map

$$\sigma_4(D) : \Lambda^1 \longrightarrow Hom_{SO(4)}(\Lambda^1, \Lambda^1).$$

This polarizes to a bundle homomorphism from the symmetric 4-tensors:

$$\begin{aligned} \tilde{\sigma}_4(D) : V(4) \oplus V(2) \oplus V(0) &\longrightarrow V(1) \otimes V(1) \\ &\cong_{SO(4)} V(0) \oplus V(1,-1) \oplus V(1,1) \oplus V(2). \end{aligned} \tag{4.2}$$

The space of these is 2-dimensional (the left and right sides of (4.2) have the summands $V(0)$ and $V(2)$ in common), and is clearly generated by $\tilde{\sigma}_4(\delta d\delta d)$ and $\tilde{\sigma}_4(d\delta d\delta)$. ■

LEMMA 4.2. *Any $D_{4,1}$ has conformal bidegree (-1,3).*

PROOF: By Lemma 4.1 and (3.7),

$$D_{4,1} = \alpha\delta d\delta d + \beta d\delta d\delta + E_2\,,$$

where $\alpha, \beta \in \mathbf{R}$ and $E_2$ is an $SO(4)$-differential operator scheme of order at most 2. Being conformally covariant and of level 4, $D_{4,1}$ has a conformal bidegree $(a, a+4)$:

$$\dot{D}_{4,1} = -4\omega D_{4,1} + a[D_{4,1}, \omega].$$

An easy calculation with (1.2) and the definition of the formal adjoint shows that $D^*_{4,1}$ has conformal bidegree $(-a-2,-a+2)$:

$$(D^*_{4,1})^\cdot = -4\omega D^*_{4,1} - (a+2)[D^*_{4,1},\omega].$$

Since $\delta d\delta d$ and $d\delta d\delta$ are formally self-adjoint, the difference of these last two equations is

$$\begin{aligned}(E_2 - E_2^*)^\cdot = &- 4\omega(E_2 - E_2^*)\\ &+ 2(a+1)[\alpha\delta d\delta d + \beta d\delta d\delta,\omega]\\ &+ a[E_2,\omega] + (a+2)[E_2^*,\omega].\end{aligned} \tag{4.3}$$

All terms here have order at most 3, and in fact, taking third order symbols,

$$0 = 2(a+1)\sigma_3([\alpha\delta d\delta d + \beta d\delta d\delta,\omega]).$$

Now if $\varepsilon$ and $\iota$ denote exterior and interior multiplication,

$$\begin{aligned}\sigma_1(d)(\xi) &= \sqrt{-1}\,\varepsilon(\xi)\,, &\quad \sigma_1(\delta)(\xi) &= -\sqrt{-1}\,\iota(\xi)\,,\\ [d,\omega] &= \varepsilon(d\omega)\,, & [\delta,\omega] &= -\iota(d\omega)\,.\end{aligned} \tag{4.4}$$

Thus

$$\begin{aligned}0 &= (a+1)\sigma_3([\alpha\delta d\delta d + \beta d\delta d\delta,\omega])(d\omega)\\ &= 4(a+1)\sqrt{-1}\{\alpha\iota(d\omega)\varepsilon(d\omega)\iota(d\omega)\varepsilon(d\omega)\\ &\quad + \beta\varepsilon(d\omega)\iota(d\omega)\varepsilon(d\omega)\iota(d\omega)\};\end{aligned}$$

since $\omega$ is arbitrary in $C^\infty(M)$ and $\alpha$ and $\beta$ are not both zero, we must have $a=-1$. ■

LEMMA 4.3. *If there is a $D_{4,1}$, then there is a formally self-adjoint $D_{4,1}$.*

PROOF: Given $a=-1$, (4.3) reads

$$(E_2 - E_2^*)^\cdot = -4\omega(E_2 - E_2^*) - [E_2 - E_2^*,\omega];$$

that is, the formally skew-adjoint part of $D_{4,1}$ is separately conformally covariant of bidegree $(-1,3)$. ■

LEMMA 4.4. *If there is a $D_{4,1}$, then there is an $O(4)$-equivariant $D_{4,1}$.*

PROOF: The orientation-reversed version $\tilde{D}_{4,1}$ of $D_{4,1}$ is also conformally covariant of bidegree $(-1,3)$. Since $\delta d\delta d$ and $d\delta d\delta$ are $O(4)$-equivariant, $D_{4,1} + \tilde{D}_{4,1}$ is properly fourth-order. ■

LEMMA 4.5. *Up to a constant factor, any* $D_{4,1}$ *must be* $3\delta d\delta d - d\delta d\delta$ *modulo second-order operator schemes.*

PROOF: By [**B1**, equation (2.1)],

$$
\begin{aligned}
&(\delta d\delta d)^{\cdot}+4\omega\delta d\delta d+[\delta d\delta d,\omega]= \\
&\qquad -\delta d(-\delta\varepsilon(d\omega)+\iota(d\omega)d) \\
&\qquad -(\delta\varepsilon(d\omega)-\iota(d\omega)d)\delta d, \\
&(d\delta d\delta)^{\cdot}+4\omega d\delta d\delta+[d\delta d\delta,\omega]= \\
&\qquad -d\delta(3d\iota(d\omega)+\varepsilon(d\omega)\delta) \\
&\qquad -(d\iota(d\omega)+3\varepsilon(d\omega)\delta)d\delta.
\end{aligned}
\tag{4.5}
$$

(In general in [**B1**], the definition of $D'^{(a)}$ gives us $-(\dot{D}+(\mathit{level}\ D)\omega D-a[D,\omega])$.) Thus by (4.4), if $E_4=\alpha\delta d\delta d+\beta d\delta d\delta$,

$$
\begin{aligned}
&\sigma_3(\dot{E}_4+4\omega E_4+[E_4,\omega])(\xi)= \\
&\sqrt{-1}\,\alpha\{-\iota(\xi)\varepsilon(\xi)\iota(\xi)\varepsilon(d\omega)-\iota(\xi)\varepsilon(\xi)\iota(d\omega)\varepsilon(\xi) \\
&\qquad +\iota(\xi)\varepsilon(d\omega)\iota(\xi)\varepsilon(\xi)+\iota(d\omega)\varepsilon(\xi)\iota(\xi)\varepsilon(\xi)\} \\
&+\sqrt{-1}\,\beta\{-3\varepsilon(\xi)\iota(\xi)\varepsilon(\xi)\iota(d\omega)+\varepsilon(\xi)\iota(\xi)\varepsilon(d\omega)\iota(\xi) \\
&\qquad -\varepsilon(\xi)\iota(d\omega)\varepsilon(\xi)\iota(\xi)+3\varepsilon(d\omega)\iota(\xi)\varepsilon(\xi)\iota(\xi)\}.
\end{aligned}
\tag{4.6}
$$

By the identities

$$
\begin{aligned}
\iota(\xi)\varepsilon(\eta)+\varepsilon(\eta)\iota(\xi)&=\xi^\alpha\eta_\alpha, \\
\varepsilon(\xi)\varepsilon(\eta)&=-\varepsilon(\eta)\varepsilon(\xi), \\
\iota(\xi)\iota(\eta)&=-\iota(\eta)\iota(\xi),
\end{aligned}
$$

choosing $\xi\perp d\omega$ in (4.6) gives

$$
\begin{aligned}
&\sigma_3(\dot{E}_4+4\omega E_4+[E_4,\omega])(\xi)= \\
&\quad\sqrt{-1}\,|\xi|^2(\alpha+3\beta)(\varepsilon(d\omega)\iota(\xi)-\varepsilon(\xi)\iota(d\omega)),
\end{aligned}
$$

which carries $\xi\mapsto\sqrt{-1}\,|\xi|^2(\alpha+3\beta)d\omega$, $d\omega\mapsto-\sqrt{-1}\,|\xi|^2(\alpha+3\beta)\xi$, and $\eta\mapsto 0$ if $\eta\perp d\omega,\xi$. Since we can only compensate the conformal non-covariance of $E_4$ with operators of order 2 and lower (recall (3.7)), we must have $\alpha+3\beta=0$. ∎

We are now reduced to proving the non-existence of a formally self-adjoint $O(4)$-differential operator of conformal bidegree $(-1,3)$ with leading term $3\delta d\delta d-d\delta d\delta$.

If $A=(A^\alpha{}_\beta)$ is a $\binom{1}{1}$ tensor, we abbreviate by $A\#$ the operator $A^\alpha{}_\beta\varepsilon(dx^\beta)\iota(\partial_\alpha)$ on differential forms.

In what follows, we shall repeatedly use the tensor product decompositions (3.3) without explicit reference.

LEMMA 4.6. *Modulo lower-order operators, the space of second-order $O(4)$-differential operator schemes on locally conformally flat manifolds is 6-dimensional, generated by $U_1 = J\delta d$, $U_2 = Jd\delta$, $U_3 = V\#\delta d$, $U_4 = V\#d\delta$, $U_5 = \delta V\#d$, $U_6 = d\delta V\#$.*

PROOF: Locally conformally flat manifolds are exactly those for which $C$, the $V(2,2)$ component of curvature, vanishes. By (3.7), we need to enumerate schemes of the form $r\nabla\nabla$ modulo those of the forms $(\nabla r)\nabla$, $(\nabla\nabla r)$, $rr$. After polarization, the second-order symbol of an $r\nabla\nabla$ thus corresponds to an element of

$$Hom_{O(4)}(V(0) \oplus V(2)) \otimes (V(0) \oplus V(2)) \otimes V(1), V(1)).$$

(The $V(1,1)$ part of $V(1) \otimes V(1)$ contributes only a curvature term to $\nabla\nabla$.) Now

$$dim\ Hom_{O(4)}(V(0) \otimes V(0) \otimes V(1), V(1) = 1\,,$$

and the corresponding operator is $\tilde{U}_1 = J\Delta$, where $\Delta = \delta d + d\delta$. (Modulo lower-order terms, we need not distinguish between $\Delta$ and $\nabla^*\nabla$.) Furthermore,

$$\begin{aligned} &dim\ Hom_{O(4)}(V(0) \otimes V(2) \otimes V(1), V(1)) = \\ &dim\ Hom_{O(4)}(V(2) \otimes V(0) \otimes V(1), V(1)) = \\ &dim\ Hom_{O(4)}(V(2), V(1) \otimes V(1)) = 1, \end{aligned}$$

the corresponding operators being

$$\begin{aligned} &(\phi_\alpha) \mapsto (\tfrac{1}{2}J(\nabla_\alpha\nabla^\beta + \nabla^\beta\nabla_\alpha - \tfrac{1}{2}\delta^\beta{}_\alpha\nabla^\lambda\nabla_\lambda)\phi_\beta), \\ &((\phi_\alpha) \mapsto (T^\beta{}_\alpha(\Delta\phi)_\beta)) = T\#\Delta. \end{aligned}$$

Modulo $rr$ terms and $J\Delta$, the first of these is $-1$ times

$$\tilde{U}_2 = Jd\delta,$$

and modulo $J\Delta$, the second is

$$\tilde{U}_3 = V\#\Delta.$$

Finally,

$$\begin{aligned} &dim\ Hom_{O(4)}(V(2) \otimes V(2) \otimes V(1), V(1)) = \\ &\qquad dim\ End_{O(4)}(V(2) \otimes V(1)) = 3, \end{aligned}$$

the operators taking $(\phi_\alpha)$ to

$$(T^{\beta\gamma}H_{\beta\gamma}\phi_\alpha),\ (T^\beta{}_\alpha H_\beta{}^\gamma\phi_\gamma),\ (T^{\beta\gamma}H_{\beta\alpha}\phi_\gamma),$$

where

$$H_{\alpha\beta}=\tfrac{1}{2}(\nabla_\alpha\nabla_\beta+\nabla_\beta\nabla_\alpha-\tfrac{1}{2}g_{\alpha\beta}\nabla^\lambda\nabla_\lambda).$$

Modulo lower-order terms and $\tilde U_1$, $\tilde U_2$, $\tilde U_3$, these last three operators are

$$\begin{aligned}\tilde U_4 &: (\phi_\alpha)\mapsto(V^\beta{}_\alpha\nabla_\beta\nabla^\gamma\phi_\gamma),\\ \tilde U_5 &: (\phi_\alpha)\mapsto(V^{\beta\gamma}\nabla_\beta\nabla_\gamma\phi_\alpha),\\ \tilde U_6 &: (\phi_\alpha)\mapsto(V^{\beta\gamma}\nabla_\beta\nabla_\alpha\phi_\gamma).\end{aligned}$$

But $\tilde U_4=-V\#d\delta$, and if $\doteq$ denotes equality modulo lower-order terms,

$$\begin{aligned}(\delta V\#d\phi)_\alpha &= -\nabla^\lambda(V\#d\phi)_{\lambda\alpha}\\ &= -\nabla^\lambda(V^\mu{}_\lambda(d\phi)_{\mu\alpha}+V^\mu{}_\alpha(d\phi)_{\lambda\mu})\\ &\doteq -V^\mu{}_\lambda\nabla^\lambda\nabla_\mu\phi_\alpha+V^\mu{}_\lambda\nabla^\lambda\nabla_\alpha\phi_\mu\\ &\quad -V^\mu{}_\alpha\nabla^\lambda\nabla_\lambda\phi_\mu+V^\mu{}_\alpha\nabla^\lambda\nabla_\mu\phi_\lambda;\end{aligned}$$

that is, $\delta V\#d\doteq-\tilde U_5+\tilde U_6+\tilde U_3+\tilde U_4$. Furthermore,

$$(d\delta V\#\phi)_\alpha=-\nabla_\alpha\nabla^\lambda(V^\beta{}_\lambda\phi_\beta),$$

so $d\delta V\#\doteq-\tilde U_6$. If $\tilde U_1,\dots,\tilde U_6$ are linearly independent, then so are $U_1,\dots,U_6$.

But by [**B1**, equations (1.27), (1.28), (1.30), (1.32)], putting $U_i^!=\dot U_i+4\omega U_i+[U_i,\omega]$,

$$\text{(4.7)}\qquad \begin{aligned}U_1^! &\doteq(\Delta\omega)\delta d,\\ U_2^! &\doteq(\Delta\omega)d\delta,\\ U_3^! &\doteq-(Hess\ \omega)\#\delta d,\\ U_4^! &\doteq-(Hess\ \omega)\#d\delta,\\ U_5^! &\doteq-\delta(Hess\ \omega)\#d,\\ U_6^! &\doteq-d\delta(Hess\ \omega)\#.\end{aligned}$$

To show that these are linearly independent conformal variation schemes, pick $x\in M$ and an orthonormal basis $(e^\alpha)$ of $M_x^*$. $Hess\ \omega$ can be arbitrarily prescribed within the symmetric 2-tensors at $x$. Suppose we have a linear relation $\Sigma_1^6 a_i \sigma_2(U_i')(\xi)\eta = 0$ for all $\xi, \eta \in M_x^*$. Then choosing $\xi = \eta = e^1$, the choice $(Hess\ \omega)_x = e^2 \otimes e^2$ gives $a_2 = 0$, the choice $(Hess\ \omega)_x = e^1 \otimes e^2 + e^2 \otimes e^1$ gives $a_4 = 0$, and the choice $(Hess\ \omega)_x = e^1 \otimes e^1 - e^2 \otimes e^2$ gives $a_4 + a_6 = 0$. Choosing $\xi = e^1$, $\eta = e^2$, the choice $(Hess\ \omega)_x = e^1 \otimes e^1 - e^2 \otimes e^2$ gives $a_3 = 0$, and the choice $(Hess\ \omega)_x = e^1 \otimes e^1 - e^3 \otimes e^3$ gives $a_5 = 0$. In particular, the second-order symbols of $U_1, \dots, U_6$ are linearly independent. ∎

The next lemma states that only one linear combination of $U_1, \dots, U_6$ can be used as the second-order correction $E_2$ to the fourth-order term given in Lemma 4.5.

LEMMA 4.7. *Modulo first-order operator schemes, any $O(4)$-equivariant $D_{4,1}$ must be a constant multiple of $3\delta d\delta d - d\delta d\delta + E_2$, where $E_2 = -4U_2 + 12U_3 + 12U_4 - 12U_5$, on locally conformally flat manifolds.*

PROOF: By (4.5) and [**B1**, equation (1.38)], with notation as in the proof of Lemma 4.6,

$$(3\delta d\delta d - d\delta d\delta)' = 3(\Delta P + P\Delta) - 6\delta P d - 2dP\delta, \tag{4.7}$$

where

$$P = \Delta\omega + 2(Hess\ \omega)\#.$$

But

$$\begin{aligned} &\delta d(\Delta\omega) \doteq \delta(\Delta\omega)d \doteq U_1', \\ &d\delta(\Delta\omega) \doteq d(\Delta\omega)\delta \doteq U_2', \\ &(\delta d(Hess\ \omega)\#\phi)_\alpha = \\ &-\nabla^\lambda(\nabla_\lambda(H^\beta{}_\alpha\phi_\beta) - \nabla_\alpha(H^\beta{}_\lambda\phi_\beta)) \\ &\doteq ((U_6' - U_3' - U_4')\phi)_\alpha, \end{aligned} \tag{4.8}$$

using $H^\beta{}_\alpha$ as an abbreviation for $\nabla_\alpha\nabla^\beta\omega$. This makes the right side of (4.7)

$$4U_2' - 12U_3' - 12U_4' + 12U_5'$$

modulo first-order schemes. But we showed in the proof of Lemma 4.6 that $U_1', \cdots, U_6'$ are linearly independent. ∎

Calculations like those in (4.7), with $-J$ in place of $\Delta\omega$ and $V$ in place of $Hess\ \omega$, now show that our prospective $D_{4,1}$, modulo first order, must agree with the $E_4 + E_2$ of [**B1**, Sec. 2.a] on locally conformally flat manifolds. But [**B1**, equation (2.2) and the equation after (2.4)] shows that $(E_4 + E_2)'$ has order zero. Thus any first-order correction $E_1$ of the schematic form $(\nabla R)\nabla$ (recall (3.7)) must have $\sigma_1(E_1') = 0$ in the locally conformally flat case. We would like to show that this forces $\sigma_1(E_1)$ itself to vanish in the locally conformally flat case.

LEMMA 4.8. *Modulo zeroth-order operators, the first-order $O(4)$-differential operator schemes on locally conformally flat 4-manifolds are 6-dimensional. A basis is given in (4.11) below.*

PROOF: By (3.4) and the remarks immediately following, the number of properly first-order operators is at most

$$\text{(4.9)} \quad \mathit{dim}\ \mathit{Hom}_{O(4)}((V(1) \oplus V(2,1) \oplus V(3)) \otimes V(1) \otimes V(1), V(1)).$$

But $\mathit{dim}\ \mathit{Hom}_{O(4)}(V(1) \otimes V(1) \otimes V(1), V(1)) = \mathit{dim}\ \mathit{End}_{O(4)}(V(1) \otimes V(1)) = 3$. By the remarks following (3.4), curvature contributions to these operators involve only $\nabla J$; the operators are

$$\begin{aligned}
\tilde{S}_1 &= (\nabla^\lambda J)\nabla_\lambda : (\phi_\alpha) \longmapsto ((\nabla^\lambda J)\nabla_\lambda \phi_\alpha), \\
\tilde{S}_2 &: (\phi_\alpha) \longmapsto ((\nabla^\lambda J)\nabla_\alpha \phi_\lambda), \\
\tilde{S}_3 &= -\varepsilon(dJ)\delta : (\phi_\alpha) \longmapsto ((\nabla_\alpha J)\nabla^\lambda \phi_\lambda).
\end{aligned}$$

From the $V(3)$ summand, we get

$$\mathit{dim}\ \mathit{Hom}_{O(4)}(V(3) \otimes V(1) \otimes V(1), V(1)) = 1$$

operator. This involves the $V(3)$ component $A$ of $\nabla T$; by (3.5) and the Bianchi identity $\nabla^\lambda T_{\lambda\alpha} = \frac{3}{4}\nabla_\alpha J$,

$$A_{\alpha\beta\gamma} = \tfrac{1}{3}(\nabla_\alpha T_{\beta\gamma} + \nabla_\beta T_{\alpha\gamma} + \nabla_\gamma T_{\alpha\beta}) - \tfrac{1}{12}(g_{\alpha\beta}\nabla_\gamma J + g_{\alpha\gamma}\nabla_\beta J + g_{\beta\gamma}\nabla_\alpha J).$$

The operator is

$$\tilde{S}_4 : (\phi_\alpha) \longmapsto (A_\alpha{}^{\beta\gamma}\nabla_\beta \phi_\gamma).$$

Finally, the $V(2,1)$ summand in (4.9) contributes

$$\begin{aligned}
&\mathit{dim}\ \mathit{Hom}_{O(4)}(V(2,1) \otimes V(1) \otimes V(1), V(1)) \\
&\quad = \mathit{dim}\ \mathit{Hom}_{O(4)}(V(1,1) \oplus V(2) \oplus V(2,2) \oplus V(3,1), \\
&\qquad V(0) \oplus V(1,1) \oplus V(2)) \\
&\quad = 2
\end{aligned}$$

operators. These involve the $V(2,1)$ component $B$ of $\nabla T$, where (by (3.6) and the Bianchi identities)

$$\text{(4.10)} \quad \begin{aligned}
B_{\alpha\beta\gamma} &= \tfrac{1}{3}(2\nabla_\alpha T_{\beta\gamma} - \nabla_\beta T_{\alpha\gamma} - \nabla_\gamma T_{\alpha\beta}) \\
&\qquad - \tfrac{1}{12}(g_{\alpha\beta}\nabla_\gamma J + g_{\alpha\gamma}\nabla_\beta J - 2g_{\beta\gamma}\nabla_\alpha J) \\
&= \tfrac{1}{3}(2\nabla_\alpha V_{\beta\gamma} - \nabla_\beta V_{\alpha\gamma} - \nabla_\gamma V_{\alpha\beta}).
\end{aligned}$$

$B$ is totally trace-free, is symmetric in the last 2 indices, and has

$$B_{\alpha\beta\gamma} + B_{\beta\gamma\alpha} + B_{\gamma\alpha\beta} = 0,$$

so the 2 operators are

$$\begin{aligned}\tilde{S}_5 &: (\phi_\alpha) \longmapsto B^{\lambda\mu}{}_\alpha \nabla_\lambda \phi_\mu, \\ \tilde{S}_6 &: (\phi_\alpha) \longmapsto B_\alpha{}^{\lambda\mu} \nabla_\lambda \phi_\mu.\end{aligned}$$

To get these operators in a form more suited to the conformal variational point of view, we introduce the *gradients*

$$\begin{aligned}G_{(2),(3)} &= Proj_{V(3)} \nabla : C^\infty(V(2)) \longrightarrow C^\infty(V(3)), \\ G_{(2),(2,1)} &= Proj_{V(2,1)} \nabla : C^\infty(V(2)) \longrightarrow C^\infty(V(2,1)), \\ G_{(1),(2)} &= Proj_{V(2)} \nabla : C^\infty(V(1)) \longrightarrow C^\infty(V(2)).\end{aligned}$$

According to [F] , all such operators are conformally covariant. We put

$$\begin{aligned}S_1 &= \iota(dJ)d, \\ (S_2\phi)_\alpha &= (\nabla^\lambda J)(G_{(1),(2)}\phi)_{\lambda\alpha}, \\ S_3 &= \varepsilon(dJ)\delta, \\ (S_4\phi)_\alpha &= A^{\lambda\mu}{}_\alpha (G_{(1),(2)}\phi)_{\lambda\mu}, \\ (S_5\phi)_\alpha &= B^{\lambda\mu}{}_\alpha (d\phi)_{\lambda\mu}, \\ (S_6\phi)_\alpha &= B^{\lambda\mu}{}_\alpha (G_{(1),(2)}\phi)_{\lambda\mu}.\end{aligned} \tag{4.11}$$

By the conformal covariance of the gradients and the fact that

$$A = G_{(2),(3)}T, \quad B = G_{(2),(2,1)}T,$$

the conformal variations of the $S_i$ have the leading symbols

$$\begin{aligned}\sigma_1(S_1')(\xi) &= \sqrt{-1}\,\iota(d\Delta\omega)\varepsilon(\xi), \\ (\sigma_1(S_2')(\xi)\phi)_\alpha &= \sqrt{-1}(\nabla^\lambda \Delta\omega)(\zeta_{(1),(2)}(\xi)\phi)_{\lambda\alpha}, \\ \sigma_1(S_3')(\xi) &= -\sqrt{-1}\,\varepsilon(d\Delta\omega)\iota(\xi), \\ (\sigma_1(S_4')(\xi)\phi)_\alpha &= \sqrt{-1}(G_{(2),(3)}F)^{\lambda\mu}{}_\alpha (\zeta_{(1),(2)}(\xi)\phi)_{\lambda\mu}, \\ (\sigma_1(S_5')(\xi)\phi)_\alpha &= \sqrt{-1}(G_{(2),(2,1)}F)^{\lambda\mu}{}_\alpha (\epsilon(\xi)\phi)_{\lambda\mu}, \\ (\sigma_1(S_6')(\xi)\phi)_\alpha &= \sqrt{-1}(G_{(2),(2,1)}F)^{\lambda\mu}{}_\alpha (\zeta_{(1),(2,1)}(\xi)\phi)_{\lambda\mu},\end{aligned}$$

where $F$ is the trace-free Hessian of $\omega$, $\sqrt{-1}\zeta_{(1),(2,1)} = \sigma_1(G_{(1),(2,1)})$, and similarly for the other gradients.

We shall show that the $S_i$ are linearly independent by showing that the $\sigma_1(S_i')$ are. Assume a linear relation $\Sigma_1^6 a_i\sigma_1(S_i') = 0$, and recall the notation of the proof of Lemma 4.6. On a flat manifold, the third covariant derivative of $\omega$ is symmetric, so $\sigma_1(S_5')$ and $\sigma_1(S_6')$ vanish. Within the symmetric 3-tensors $V(1) \oplus V(3)$, $\nabla\nabla\nabla\omega$ is arbitrarily prescribable at a point; making a choice with vanishing $V(1)$ component, we get $a_4 = 0$. Making a choice with vanishing $V(3)$ component leaves us with $\Sigma_1^3 a_i\sigma_1(S_i')$. But $(\sigma_1(S_i')(\xi)\phi)_x$ for $i = 1, 2$, or $3$ involves respectively the $V(1,1), V(2)$, or $V(0)$ component of $\xi \otimes \phi_x$. Choosing only one of these to be nonzero at a time, we see that $a_1 = a_2 = a_3 = 0$.

To show that $a_5$ and $a_6$ must vanish, we work with a nonflat, but still conformally flat metric. By (3.6),

$$\begin{aligned}(G_{(2),(2,1)}F)_{\alpha\beta\gamma} &= \tfrac{1}{3}(2\nabla_\alpha F_{\beta\gamma} - \nabla_\beta F_{\alpha\gamma} - \nabla_\gamma F_{\alpha\beta}) \\ &\quad - \tfrac{1}{9}(g_{\alpha\beta}\nabla^\lambda F_{\lambda\gamma} + g_{\alpha\gamma}\nabla^\lambda F_{\lambda\beta} - 2g_{\beta\gamma}\nabla^\lambda F_{\lambda\alpha}).\end{aligned}$$

By the Ricci identities and conformal flatness,

$$\begin{aligned}\nabla_\beta\nabla_\alpha\nabla_\gamma\omega &= \nabla_\alpha\nabla_\beta\nabla_\gamma\omega - R^\mu{}_{\gamma\beta\alpha}\nabla_\mu\omega \\ &= \nabla_\alpha\nabla_\beta\nabla_\gamma\omega + V_{\beta\gamma}\nabla_\alpha\omega - V_{\alpha\gamma}\nabla_\beta\omega \\ &\quad + V_\alpha{}^\mu g_{\gamma\beta}\nabla_\mu\omega - V_\beta{}^\mu g_{\gamma\alpha}\nabla_\mu\omega;\end{aligned}$$

in particular, using the symmetry of the Hessian,

$$\begin{aligned}\nabla^\lambda\nabla_\lambda\nabla_\alpha\omega &= \nabla^\lambda\nabla_\alpha\nabla_\lambda\omega \\ &= \nabla_\alpha\nabla^\lambda\nabla_\lambda\omega + 2V^\lambda{}_\alpha\nabla_\lambda\omega + J\nabla_\alpha\omega.\end{aligned}$$

Thus in the conformally flat case,

$$\begin{aligned}(G_{(2),(2,1)}F)_{\alpha\beta\gamma} &= \tfrac{1}{9}(g_{\alpha\gamma}V^\lambda{}_\beta + g_{\alpha\beta}V^\lambda{}_\gamma - 2g_{\beta\gamma}V^\lambda{}_\alpha)\nabla_\lambda\omega \\ &\quad + \tfrac{1}{3}(V_{\alpha\gamma}\nabla_\beta\omega + V_{\alpha\beta}\nabla_\gamma\omega - 2V_{\beta\gamma}\nabla_\alpha\omega) \\ &\quad - \tfrac{1}{9}J(g_{\alpha\gamma}\nabla_\beta\omega + g_{\alpha\beta}\nabla_\gamma\omega - 2g_{\beta\gamma}\nabla_\alpha\omega),\end{aligned}$$

so that

$$\begin{aligned}-\sqrt{-1}(\sigma_1(S_5')(\xi)\phi)_\gamma &= \tfrac{1}{3}(\xi\wedge\phi)_{\alpha\gamma}(J\nabla^\alpha\omega - V^{\alpha\beta}\nabla_\beta\omega) \\ &\quad - (\xi\wedge\phi)_{\alpha\beta}V^\beta{}_\gamma\nabla^\alpha\omega, \\ -\sqrt{-1}(\sigma_1(S_6')(\xi)\phi)_\gamma &= \tfrac{1}{3}(\zeta_{(1),(2,1)}(\xi)\phi)_{\alpha\beta}(V^{\alpha\beta}\nabla_\gamma\omega - V^\beta{}_\gamma\nabla^\alpha\omega) \\ &\quad + \tfrac{1}{9}(\zeta_{(1),(2,1)}(\xi)\phi)_{\alpha\gamma}(J\nabla^\alpha\omega - V^{\alpha\beta}\nabla_\beta\omega).\end{aligned}$$

By, e.g., [**LP**, (2.5) and (2.6)], the value of $V$ can be arbitrarily prescribed within the symmetric 2-tensors at a given point $x$ without disturbing conformal flatness. In the notation of the proof of Lemma 4.6, the choice $(\nabla\omega)_x = \xi = e^1, \phi_x = e^2$, and $V_x = e^1 \otimes e^2 + e^2 \otimes e^1$ gives

$$-\sqrt{-1}(\sigma_1(S_5')(\xi)\phi)_x = \tfrac{2}{3}e^1,$$
$$-\sqrt{-1}(\sigma_1(S_6')(\xi)\phi)_x = \tfrac{2}{9}e^1,$$

while the same choice with $V_x = e^1 \otimes e^1$ gives

$$-\sqrt{-1}(\sigma_1(S_5')(\xi)\phi)_x = 0,$$
$$-\sqrt{-1}(\sigma_1(S_6')(\xi)\phi)_x = -\tfrac{1}{3}e^2.$$

This shows that $a_5 = a_6 = 0$. ∎

As a corollary to the proof of Lemma 4.8, we get that any $D_{4,1}$ must agree with the operator given in Lemma 4.7 down to *zeroth-* (not just first-) order operators in the conformally flat case.

LEMMA 4.9. *Modulo zeroth-order operators, any $O(4)$-equivariant $D_{4,1}$ must be a constant multiple of $3\delta d\delta d - d\delta d\delta + E_2$, where $E_2$ is as in Lemma 4.7, on locally conformally flat manifolds.*

PROOF: By Lemmas 4.7 and 4.8,

$$D_{4,1} = 3\delta d\delta d - d\delta d\delta + E_2 + \Sigma_1^6 a_i S_i \quad (\textit{modulo zeroth order})$$

for some constants $a_i$. By [**B1**, equation (2.2) and Lemma 2.2], $(3\delta d\delta d - d\delta d\delta + E_2)'$ has order 0. Thus by the proof of Lemma 4.8, all $a_i$ vanish. ∎

In fact, [**B1**, equation (2.2) and Lemma 2.2] show that

$$\begin{aligned}(3\delta d\delta d - d\delta d\delta + E_2)' = 3(Z^2)' - 6\{(|V|^2)' - 2(V^{\cdot}{}_{\mu}V^{\mu}{}_{\cdot})'\# \\ + 2(\nabla^{\lambda}\omega)(2\nabla_{\lambda}V^{\cdot}{}_{\cdot} - \nabla_{\cdot}V^{\cdot}{}_{\lambda} - \nabla^{\cdot}V_{\cdot\lambda})\#\},\end{aligned}$$

where $Z = J - 2V\#$ and $|V|^2 = V^{\alpha}{}_{\beta}V^{\beta}{}_{\alpha}$. Thus we can only have a $D_{4,1}$ if there is a zeroth-order operator $P$ for which

$$\begin{aligned}P' &\stackrel{?}{=} (\nabla^{\lambda}\omega)(2\nabla_{\lambda}V^{\cdot}{}_{\cdot} - \nabla_{\cdot}V^{\cdot}{}_{\lambda} - \nabla^{\cdot}V_{\cdot\lambda})\# \\ &= 3(\nabla^{\lambda}\omega)B_{\lambda}{}^{\cdot}{}_{\cdot}\#\end{aligned}$$

(recall (4.10)) on conformally flat manifolds.

LEMMA 4.10. *Modulo actions of $r \otimes r$, the zeroth-order $O(4)$-operator schemes on 1-forms in a conformally flat Riemannian 4-manifold are 2-dimensional. A basis is given by $\Delta J$, $(trace - free\ Hessian\ J)\#$.*

PROOF: By (3.7), the operator schemes in question are actions of $\nabla\nabla r$. There are 2 possible ways to enumerate these: first, using (2.4), we note that such an operator corresponds to an element of

$$Hom_{O(4)}(V(1) \otimes (V(1) \oplus V(2,1) \oplus V(3)) \otimes V(1), V(1)).$$

This approach gives 6 potential operators; however, it ignores the fact that $\nabla r$ is not just an arbitrary section of $V(1) \oplus V(2,1) \oplus V(3)$, but one in the range of $\nabla$. The other way is to temporarily ignore the second Bianchi identity and view our operator (modulo actions of $r \otimes r$) as a section of

$$Hom_{O(4)}((V(0) \oplus V(2)) \otimes (V(0) \oplus V(2)) \otimes V(1), V(1)).$$

Taking this second approach, we calculate

$$dim\ Hom(V(0) \otimes V(0) \otimes V(1), V(1)) = 1,$$

the operator being $\Delta J$;

$$\begin{aligned} &dim\ Hom(V(0) \otimes V(2) \otimes V(1), V(1)) \\ &\quad = dim\ Hom(V(2) \otimes V(0) \otimes V(1), V(1)) = 1, \end{aligned}$$

the operators being

$$(\nabla^\lambda \nabla_\lambda T^{\cdot}{}_{\cdot})\#, \quad (trace - free\ Hessian\ J)\#$$

respectively. Finally,

$$dim\ Hom(V(2) \otimes V(2) \otimes V(1), V(1)) = dim\ End(V(2) \otimes V(1)) = 3.$$

The 3 operators produced,

$$\begin{aligned} (\phi_\alpha) &\longmapsto (\nabla_\alpha \nabla^\lambda T_\lambda{}^\mu)\phi_\mu, \\ (\phi_\alpha) &\longmapsto (\nabla^\lambda \nabla^\mu T_{\lambda\alpha})\phi_\mu, \\ (\phi_\alpha) &\longmapsto (\nabla^\lambda \nabla^\mu T_{\lambda\mu})\phi_\alpha, \end{aligned}$$

are linearly independent of the above for $T$ an arbitrary section of $V(2)$, but collapse to linear combinations of the above for $T$ satisfying the second Bianchi identity. As for the 3 surviving operators, the *Bach tensor* $Y$ (see [B] , or [B1, Theorem 2.5] for a description in the present notation) agrees with $(\nabla^\lambda \nabla_\lambda T^\alpha{}_\beta)$, modulo actions of $r \otimes r$. Being a relative conformal invariant, $Y$ vanishes on conformally flat manifolds, and it is straightforward to show that the 2 remaining operators are linearly independent. ∎

THEOREM 4.11. *There is no properly fourth-order covariant of conformal change acting on 1-forms in pseudo-Riemannian 4-manifolds of any metric signature.*

PROOF: By analytic continuation in signature [**BØ**, Sec. 7] , we need only work with Riemannian 4-manifolds. By Lemmas 4.1-4.9, we need only show that there is no zeroth-order $O(4)$-operator scheme $P$ satisfying (4.12). By Lemma 4.10, all zeroth-order $O(4)$-operator schemes in the conformally flat case are linear combinations of $\Delta J$, $(trace-free\ Hessian\ J)\#$, and actions of $r \otimes r$. But by [**B1**, equation (2.9)] , denoting the trace-free Hessian operator by $H$,

$$\begin{aligned}(\Delta J)' &= -\Delta^2\omega - 2(\nabla^\lambda\omega)\nabla_\lambda J + 2(\Delta\omega)J,\\ ((HJ)')^\alpha{}_\beta &= -(H\Delta\omega)^\alpha{}_\beta\\ &\quad + 3((\nabla^\alpha\omega)\nabla_\beta J + (\nabla_\beta\omega)\nabla^\alpha J - \tfrac{1}{2}(\nabla^\lambda\omega)(\nabla_\lambda J)\delta^\alpha{}_\beta)\\ &\quad + 2(H\omega)^\alpha{}_\beta J.\end{aligned}$$

By (4.1), the conformal variations ($'$) of actions of $r \otimes r$ are actions of $r \otimes Hess\ \omega$.

Thus, since the jets of $\omega$ are prescribable at a point $x \in M$, we need only produce a conformally flat metric near $x$ with $(\nabla J)_x = 0, B_x \neq 0$. This just amounts to prescribing the $V(2,2)$ component of $R$ and the $V(1), V(2,1)$ components of $\nabla R$ independently at $x$. But a classical theorem (see, e.g., [**Ku, Be, St** ] ) states that the curvature and Bianchi identities are the only pointwise conditions on $(R, \nabla R)$; that is, that all components of $R$ and $\nabla R$ corresponding to the irreducible summands in (3.1), (3.4) are independently prescribable at a point. ∎

## REFERENCES


[**ABP**] M. ATIYAH, R. BOTT, and V. PATODI, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.

[**A**] T. AUBIN, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269–296.

[**Ba**] R. BACH, *Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs*, Math. Z. **9** (1921), 110–135.

[**Be**] A. BESSE, "Géometrie riemannienne en dimension 4," Cedic, Paris, 1981.

[**B1**] T. BRANSON, *Differential operators canonically associated with a conformal structure*, Math. Scand. **57** (1985), 293–345.

[**B2**] T. BRANSON, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987), 199–291.

[**B3**] T. BRANSON, *Differential form representations of* $O(p,q)$, in preparation.

[**BØ**] T. BRANSON and B. ØRSTED, *Conformal deformation and the heat operator*, to appear, Indiana U. Math. J. **37** (1988).

[DP] H. DONNELLY and V. PATODI, *Spectrum and the fixed point sets of isometries - II*, Topology **16** (1977), 1–11.

[F] H. FEGAN, *Conformally invariant first order differential operators*, Quart. J. Math. Oxford (2) **27** (1976), 371–378.

[H] S. HELGASON, "Differential Geometry, Lie Groups, and Symmetric Spaces," Pure and Appl. Math. 80, Academic Press, New York, 1978.

[J] H. P. JAKOBSEN, *Conformal covariants*, Publ. RIMS, Kyoto Univ. **22** (1986), 345–364.

[KN] S. KOBAYASHI and K. NOMIZU, "Foundations of Differential Geometry I, II," Interscience, New York, 1963 and 1969.

[Ko] Y. KOSMANN, *Dérivées de Lie des spineurs*, Ann. Mat. Pura Appl. (4) **XCI** (1972), 317–395.

[Ku] R. KULKARNI, *On the Bianchi identities*, Math. Ann. **199** (1972), 175–204.

[LP] J. LEE and T. PARKER, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37–91.

[Sc] R. SCHOEN, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. **20** (1984), 479–495.

[St] R. STRICHARTZ, *Linear algebra of curvature tensors and their covariant derivatives*, to appear, Canad. J. Math..

[T] N. TRUDINGER, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa **22** (1968), 265–274.

[W] H. WEYL, "The Classical Groups, their Invariants and Representations," Princeton U. Press, 1946.

[Y] H. YAMABE, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.

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