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Riemann-Roch theorem after D. Toledo and Y.-L. Tong

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Riemann-Roch theorem after D. Toledo and Y.-L. Tong[1)]

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To the memory of Vadik Knizhnik

Introduction

Local versions of Riemann-Roch type theorems attract much attention of mathematicians and, in the last time, physicists[2)]. Roughly speaking, the problem is to establish an equality between certain cohomology classes, which asserts Riemann-Roch theorem, on the level of cocycles, for example for closed differential forms representing these classes. In the remarkable series of papers [1-4] Domingo Toledo, Yue Lin Tong and Nigel O'Brian gave a local proof in Čech cohomology of absolute - RR-Hirzebruch, and relative - RR-Grothendieck-theorems in Čech cohomology.

The aim of the present paper is mainly pedagogical. In it I try to explain the Toledo-Tong's proof of the absolute RR on the first nontrivial example of surfaces.

Let X be a smooth n-dimensional complex algebraic variety, E a vector bundle on X. The Serre-Grothendieck duality theory gives a

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1) This paper is in final form and no version of it will be submitted for publication elsewhere.

2) For one of the earliest (and the best) papers on this subject, see [10]; examples of recent results are [11], [12], [5].

canonical element $\delta(E) \in H^n(X, \Omega^n_X)$ whose integral over X when X is compact is equal to the Euler characteristics $\sum_{i=0}^{n} (-1)^i \dim H^i(X, E)$. Toledo and Tong construct a canonical Čech cocycle representing $\delta(E)$ and prove that

$$(0.1) \qquad \delta(E) = (\mathrm{ch}\, E \cdot \mathrm{Td}(\mathcal{T}_X))_n$$

where $\mathcal{T}_X$ is the tangent bundle, ch the Chern character and Td the Todd genus (see §2). They do it introducing the very interesting new homological technique of so called "twisted complexes"[1)]. Unfortunately, the last step of the proof is implicit: following the idea of [10], they show that $\delta(E)$ is some polynomial of Chern classes of E and $\mathcal{T}_X$ and then use the Hirzebruch trick to show that this polynomial is equal to the r.h.s. of (0.1). In this paper are presented the explicit calculations for $n = 1, 2$.

In §1 I carry out the calculation of $\delta(E)$ in the easy case of curves when there is no need of twisted resolutions. This section may be also useful as an introduction to [5].

In §2 are recalled the necessary facts from Grothendieck duality theory. In §3 is explained the theory of twisted complexes.

In §4, which is the heart of the paper, I calculate directly $\delta(\mathcal{O}_X)$ for surfaces. This calculation allows to formulate a certain amusing conjecture 4.8.1 which says roughly speaking that different higher homotopies appearing in the twisted Koszul-Toledo-Tong resolution of the diagonal give rise to different summands in the exp-

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1) Note that one of the key points of this theory - a theorem that every coherent sheaf has a twisted resolution by locally free ones, cf. 3.2.3.1, 3.3.4.1, - appears later (in an equivalent form) in [7], 3.2.9.

ression of Todd genus through Chern character.

Finally, in §5 I explain briefly how the previous technique may be used for generalisation of some results of [5] to higher dimensions.

This work is partly based on lectures given at Winter school "Geometry and physics" held in Srni (Čechia), January 1988. I am very grateful to its organizers, and especially to Vladimir and Jiří Součeks for their hospitality during my stay there.

Notations. If X, Y are varieties, E (resp. F) - a sheaf on X (resp. Y) then we put $E \boxtimes F := p_1^* E \otimes p_2^* F$, where $p_1 : X \times Y \longrightarrow X$, $p_2 : X \times Y \longrightarrow Y$ are the projections.

For a ring A $\mathfrak{gl}_m(A) = \mathrm{Mat}_m(A)$ denotes algebra of $m \times m$-matrices with coefficients in A.

Symbol ∎ means the end of a proof or the absence of it.

§1. Curves

Let X be a smooth compact complex algebraic curve, E a vector bundle of rank m over X. Let $\omega = \Omega^1_X$ denote the sheaf of holomorphic differentials on X, $\int_X : H^1(X, \omega) \xrightarrow{\sim} \mathbf{C}$. Following [1] put $E' = E^\vee \otimes \omega$ where $E^\vee = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$ denotes the dual vector bundle.

We also use the notation ω^k for k-th tensor power of ω (for $k < 0$ $\omega^k = \mathcal{T}^{\otimes -k}$, $\mathcal{T} = \mathcal{T}_X = \omega^\vee$ = the tangent bundle of X).

1.1. The sheaf $\mathcal{P}(E)$. Consider the sheaf $E \boxtimes E'$ on $X \times X$. By Künneth formula and Serre duality we have

$$(1.1.1) \quad \begin{aligned} H^1(X \times X, E \boxtimes E') &= \sum_{i=0}^{1} H^i(X, E) \otimes H^{1-i}(X, E') = \\ &= \sum H^i(X, E) \otimes H^i(X, E)^\vee = \sum_{i=0}^{1} \mathrm{End}\, H^i(X, E) \end{aligned}$$

On the other hand, consider the sheaf $E \boxtimes E'/(E \times E')(-\Delta)$ where $\Delta : X \longrightarrow X \times X$ is the diagonal. This sheaf is equal to $\Delta_*(\mathrm{End}\, E \otimes \omega)$. Let tr_* denote the composition

$$H^1(X\times X,\ E\boxtimes E') \longrightarrow H^1(X\times X,\ E\boxtimes E'/E\boxtimes E'(-\Delta)) =$$

$$= H^1(X,\ \underline{\mathrm{End}}\ E\otimes\omega) \xrightarrow{tr} H^1(X,\omega) \xrightarrow{\int} \mathbb{C}$$

1.1.2. Lemma. For $f = (f^0, f^1) \in \mathrm{End}\ H^0(X, E) \oplus \mathrm{End}\ H^1(X, E) = H^1(X\times X,\ E\boxtimes E')$, $tr_* f = tr\ f^0 - tr\ f^1$. ■

1.1.3. Now for $a \in \mathbb{Z}$ put (cf. [5]) $\mathcal{P}(E)^{\le a} = E\boxtimes E'((a+1)\Delta)$, $\mathcal{P}(E)^{a,b} = \mathcal{P}(E)^{\le b}/\mathcal{P}(E)^{\le a}$ $(a \le b)$. All $\mathcal{P}(E)^{a,b}$ are supported on and we consider them as sheaves on X. Also put $\mathcal{P}(E) = \bigcup_a \mathcal{P}(E)^{\le a}$, $\mathcal{P}(E)^{a,\infty} = \bigcup_{b\ge a} \mathcal{P}(E)^{a,b}$. We have $\mathcal{P}(E)^{\le -1} = E\boxtimes E'$, $\mathcal{P}(E)^{-1,a} = \mathcal{D}(E)^{\le a}$:= the sheaf of differential operators $E \longrightarrow E$ of order $\le a$; $\mathcal{P}(E)^{-1,\infty} = \mathcal{D}(E) := \bigcup_a \mathcal{D}(E)^{\le a}$. We denote by sym projections $\mathcal{P}(E)^{\le a} \longrightarrow \mathcal{P}(E)^{\le a}/\mathcal{P}(E)^{\le a-1} = \mathcal{T}_X^{\otimes a}\otimes \mathrm{End}\ E$. We have the exact sequence

$$0 \longrightarrow E\boxtimes E' \longrightarrow \mathcal{P}(E) \longrightarrow \mathcal{D}(E) \longrightarrow 0 \tag{1.1.4}$$

Let $\partial : H^0(X, D(E)) \longrightarrow H^1(X, E\boxtimes E') = \sum \mathrm{End}\ H^i(X, E)$ be the corresponding coboundary map.

1.1.5. Lemma. For $D \in H^0(X, D(E))$ $\partial(D)$ is the endomorphism in cohomology induced by D. ■

1.1.6. Corollary. Let

$$0 \longrightarrow \omega \longrightarrow \tilde{D}(E) \longrightarrow D(E) \longrightarrow 0$$

be the extension induced from (1.4.4) by $E\boxtimes E' \longrightarrow \mathcal{P}(E)^{-2,-1} \xrightarrow{tr} \omega$. Then for $D \in H^0(X, D\ \ )$ $\int_X \partial(D) \in \mathbb{C}$ is equal to $tr\ D|_{H^0(X,E)} - tr\ D|_{H^1(X,E)}$.

In particular, $\int \partial(1) = \chi(X, E) := \dim H^0(X, E) - \dim H^1(X,E)$. ■

1.2. Atiyah algebras and Chern classes

Put

$$\mathcal{A}(E) = \{\partial \in \mathcal{D}(E)^{\le 1} : \mathrm{sym}(\partial) \in \mathcal{T}_X \subset \mathcal{T}_X\otimes \mathrm{End}\ E\}.$$

$\mathcal{A}(E)$ is a Lie subalgebra of $\mathcal{D}(E)$ (where for $\partial_1, \partial_2 \in \mathcal{D}(E)$ $[\partial_1, \partial_2] := \partial_1\partial_2 - \partial_2\partial_1$); in fact it is a Lie algebra of infinitesimal symmetries of (X, E). We call $\mathcal{A}(E)$ the Atiyah algebra of E, cf. [5].

We have an extension of $_X$-modules

(1.2.1) $0 \longrightarrow \mathrm{End}\, E \longrightarrow \mathcal{A}(E) \longrightarrow \mathcal{T}_X \longrightarrow 0$

Let $c(E) \in \mathrm{Ext}^1(\mathcal{T}_X, \mathrm{End}\, E) = H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$ be its class. By definition, the first Chern class (style Atiyah) of E is $c_1(E) := \mathrm{tr}\, c(E)$ where

$$\mathrm{tr}: H^1(X, \Omega^1 \otimes \mathrm{End}\, E) \longrightarrow H^1(X, \Omega^1)$$

is induced by trace $\mathrm{End}\, E \longrightarrow \mathcal{O}_X$.

If E is given by a Čech cocycle $\varphi = (\varphi_{ij}) \in \check{Z}^1(\underline{U}, GL_m(\mathcal{O}_X))$ on some open covering $\underline{U} = \{U_i\}$ of X then one easily sees that c(E) is given by cocycle $\varphi_{ij}^{-1}\, d\varphi_{ij}$, so

(1.2.2) $c_1(E) = \{\mathrm{tr}\, \varphi_{ij}^{-1}\, d\varphi_{ij} = \{(\det \varphi_{ij})^{-1}\, d(\det \varphi_{ij})\}$

1.2.3. Example. Choose a small open covering $X = \bigcup U_i$ with local coordinates x_i in U_i. Then the transition functions for $\mathcal{T}_X$ are $\dot{\alpha}_{ij}(x_i)$ where $x_j = \alpha_{ij}(x_i)$ in $U_i \cap U_j$, $\dot{} = \frac{d}{dx_i}$, so $c_i(\mathcal{T}_X)$ is represented by a cocycle $\dot{\alpha}_{ij}^{-1}\, \ddot{\alpha}_{ij}(x_i)$

1.3. Riemann-Roch.

Theorem. $\chi(X, E) = \int_X (c_1(E) + \frac{m}{2} c_1(\mathcal{T}_X))$

Proof (Toledo-Tong). Let us calculate class o(1) in 1.1.6. Choose a small open covering $X = \bigcup U_i$ with local coordinates and trivialisations of E over each U_i.

Let U be an open from U with local coordinate x, hence local coordinate (x, y) in $U \times U$. 1 is represented by $\frac{dy \cdot I_m}{y - x}$ in $U \times U$. ($I_m \in GL_m$ an identity.) Under a change of coordinates $x \longrightarrow \alpha(x)$, $y \longrightarrow \alpha(y)$ and a gauge transformation $B \in GL_m$ it transforms to

$$B(x)^{-1} \frac{d\alpha(y)}{\alpha(y) - \alpha(x)} B(y) = \frac{dy}{y - x} + \left(\frac{m}{2} \dot{\alpha}^{-1}(x) \ddot{\alpha}(x) + \operatorname{tr} B(x)^{-1} B(x)\right) dy$$

so

$$\partial(1) = \frac{m}{2} c_1(\mathcal{T}_X) + c_1(E)$$

on the level of Cech cocycles. ■

1.4. <u>Algebras $\hat{\mathcal{A}}(E)$</u>.

Put

$$\hat{\mathcal{A}}(E) = \{ \partial \in \mathcal{D}(E)^{\leq 1} / \mathcal{D}(E)^{\leq -2} : \varepsilon(\partial) \in \mathcal{T}_X \subset \mathcal{T}_X \otimes \operatorname{End} E \} / \mathcal{K}$$

where

$$\mathcal{K} := \ker(\omega \otimes \operatorname{End} E \xrightarrow{tr} \omega \longrightarrow \omega / d\mathcal{O}) \subset \omega \otimes \operatorname{End} E \subset \mathcal{D}(E)^{\leq -1} / \mathcal{D}(E)^{\leq -2}$$

We have an extension

(1.4.1) $\quad 0 \longrightarrow \omega / d\mathcal{O} \longrightarrow \hat{\mathcal{A}}(E) \xrightarrow{p} \mathcal{A}(E) \longrightarrow 0$

$\mathcal{A}(E)$ as an algebra of infinitesimal symmetries of (X, E) acts on $\hat{\mathcal{A}}(E)$ and $\mathcal{A}(E)$ with bracket $[\alpha, \beta] = p(\alpha)(\beta)$ becomes a Lie algebra - a central extension of $\mathcal{A}(E)$ by $\omega / d\mathcal{O}$. It plays a key role in relative Grothendieck-Riemann-Roch for families of curves, see [5].

§2. Duality

Let X be a smooth complete variety over $\mathbb{C}$, $\dim X = n$, $\omega_X := \Omega^n_X$, E a vector bundle of rank m over X, $i : Y \hookrightarrow X$ a smooth closed subvariety, $\dim Y = n - r$. We put $E' = E^V \otimes \omega_X$ where E^V is the dual vector bundle.

2.1. <u>Gysin map</u>. In this n° we follow the presentation of [2]. The restriction $i^*: H^j(X, E) \longrightarrow H^j(Y, i^* E)$ induces map of dual vector spaces $H^j(Y, i^* E)^V \longrightarrow H^j(X, E)^V$ which by Serre duality is the same as

$$H^{n-r-j}(Y, (i^* E)') \longrightarrow H^{n-j}(X, E').$$

In other words we obtain maps

(2.1.1) $\quad i : H^p(Y, (i^* E)') \longrightarrow H^{p+r}(X, E')$

According to Grothendieck they can also be obtained as follows, [6]. Consider the spectral sequence

$$(2.1.2) \quad H^p(X, \underline{Ext}^q_{\mathcal{O}_X}(\mathcal{O}_Y, E')) \Rightarrow Ext^{p+q}_X(\mathcal{O}_Y, E')$$

Since E is locally free, $\underline{Ext}^q_{\mathcal{O}_X}(\mathcal{O}_Y, E') = 0$ for $q \neq r$, and $\underline{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Y, E') = i_*(\Lambda^r N_Y) \otimes E' = i_*(\Lambda^r N_Y \otimes i^* E')$ where N_Y is the normal sheaf of Y in X. Now from exact sequence

$$0 \to N_Y^V \to i^*\Omega^1_X \to \Omega^1_Y \to 0$$

follows that $i^*\omega_X \cong \omega_Y \otimes \Lambda^r N_Y$, so $\Lambda^r N_Y \otimes i^*E' \cong (i E)'$. Thus we obtain isomorphisms

$$(2.1.3) \quad Res : Ext^p_X(\mathcal{O}_Y, E') \xrightarrow{\sim} H^{p-r}(X, i_*(i^*E)') = H^{p-r}(Y, (i^*E)')$$

Then i_* is just the composition

$$(2.1.4) \quad H^p(Y, (i^* E)') \xrightarrow[\sim]{Res^{-1}} Ext^{p+r}_X(\mathcal{O}_Y, E') \to Ext^{p+r}_X(\mathcal{O}_X, E') =$$

$$= H^{p+r}(X, E').$$

2.2. Let us apply the above to the diagonal embedding $\Delta: X \to X \times X$ and to the sheaf $F = E' \boxtimes E$ on $X \times X$. We have $(i^*F)' = \underline{End}\, E$, $F' = E \boxtimes E'$, so we obtain maps

$$Res^{-1} : H^p(X, \underline{End}\, E) \to Ext^{p+n}_{X \times X}(\mathcal{O}_X, E \boxtimes E')$$

In particular, we have a canonical element

$$(2.2.1) \quad Res^{-1}(1) \in Ext^n_{X \times X}(\mathcal{O}_X, E \boxtimes E')$$

Restricting it on X we obtain

$$\Delta^* Res^{-1}(1) \in H^n(X, \underline{End}\, E \otimes \omega_X) \xrightarrow{tr} H^n(X, \omega_X) \xrightarrow[\sim]{\int} \mathbb{C}$$

2.2.2. Lemma (cf. 1.1.6). $\int_X tr\, \Delta^* Res^{-1}(1) = \chi(X, E) := \sum_{i=0}^n (-1)^i \dim H^i(X, E)$. ∎

So Riemann-Roch problem is to calculate the class $\delta(1_E) :=$

$\operatorname{tr} \Delta^* \operatorname{Res}^{-1}(1)$.

(2.3) Theorem (Riemann-Roch-Hirzebruch-Grothendieck).

$$\delta(1_E) = (\operatorname{ch} E \cdot \operatorname{Td} \mathcal{T}_X)_n$$

where $\mathcal{T}_X$ is the tangent bundle, ch - Chern character, Td - Todd genus, $(\cdot)_n$ denotes n-th homogeneous component (see 4.2.3).

The case $n = 1$ was treated on §1. In 4.9 we shall prove (2.3) for $n = 2$.

2.4. More generally, one can define an n-dimensional analogue of the extension (1.1.4). To do this we use Grothendieck duality theory [6] that generalises Serre duality.

This theory asserts that on derived categories of complexes with quasicoherent cohomology there exist functors $Rf^! : D(Y) \longrightarrow D(X)$ (for $f : X \longrightarrow Y$) right adjoint to functors of direct image with compact supports $Rf_! : D(X) \longrightarrow D(Y)$ (recall that if f is proper then $Rf_! = Rf_*$), with the following properties.

(2.4.1) If f is finite then $Rf^! M = R\, \underline{\operatorname{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X, M)$.

(2.4.2) If f is smooth of relative dimension n then

$$Rf^! M = f^* M \otimes \omega_{X/Y}[n],$$

where $\omega_{X/Y} = \Omega^n_{X/Y}$ is a sheaf of relative n-differentials, [6, ch. III, §§ 2, 6].

Let

(2.4.3) $$\operatorname{Tr} = \operatorname{Tr}_f : Rf_! Rf^! \longrightarrow \operatorname{id}_{D(Y)}$$

denote the adjunction morphism.

For $0 \leq r \leq \infty$ consider the r-th infinitesimal neighbourhood of the diagonal $\Delta^{(r)} : X^{(r)} \longrightarrow X \times X$, $X^{(r)} = \operatorname{Spec} \mathcal{O}_{X \times X} / J^{r+1}$ where J is the ideal of Δ. Put $p_i^{(r)} = p_i \cdot \Delta^{(r)}$ where $p_i : X \times X \longrightarrow X$, $i = 1, 2$, are projections.

It is well known that

$$\mathcal{D}(\mathcal{O})^{\leq r} = \underline{\operatorname{Hom}}_{\mathcal{O}_X}(p^{(r)}_{1*} \mathcal{O}_{X^{(r)}}, \mathcal{O}_X),$$

and more generally, for vector bundles E, F on X

$$\mathrm{Diff}(E, F)^{\leq r} = \underline{\mathrm{Hom}}(p_{1*}^{(r)} p_2^{(r)*} E, F),$$

[9, 16.8]. Thus we have by (2.4.1), (2.4.2)

$$D(\mathcal{O})^{\leq r} = R\, p_1^{(r)!}\, \mathcal{O}_X = R\, \Delta^{(r)!}\, R\, p_1^* \,\mathcal{O}_X = R\, \Delta^{(r)!}\, p_1^* \,\omega[n].$$

(note that $p_1^* \omega = \mathcal{O} \boxtimes \omega$). So we have a trace map

$$\mathrm{Tr}_{\Delta^{(r)}} : \Delta_*^{(r)} \mathcal{D}(\mathcal{O})^{\leq r} = \Delta_*^{(r)}\, R\, \Delta^{(r)!}\, p_1^*\omega[n] \longrightarrow p_1^*\omega[n],$$

i.e. the canonical element

$$\mathrm{Tr} \in \mathrm{Ext}^n_{X\times X} (\mathcal{D}(\mathcal{O}),\ p_1^*\omega).$$

Tensoring it by $E \boxtimes E^v$ we obtain the canonical element

$$\mathrm{Tr} \in \mathrm{Ext}^n_{X\times X} (\mathcal{D}(E),\ E \boxtimes E').$$

Its Ioneda representative

(2.4.4) $$0 \longrightarrow E \boxtimes E' \longrightarrow \mathcal{T}(E)_n \longrightarrow \dots \longrightarrow \mathcal{T}(E)_o \longrightarrow \mathcal{D}(E) \longrightarrow 0$$

is analogue of (1.1.4). Unfortunately it is well defined only in the derived category. In §5 we'll show using the method of Toledo-Tong how to construct a certain canonical "twisted" extension representing this element.

Of course one has an analogue of 1.1.5:

(2.4.5) Lemma. For $D \in H^o(X, D(E))$ $\partial(D) \in H^n(X \times X,\ E \boxtimes E') = \sum_i \mathrm{End}(H^i(X, E))$, where ∂ is the coboundary operator corresponding to (2.4.4), is the endomorphism induced by D in cohomology. ∎

(2.4.6) The extension induced from (2.4.4) by

$$E \boxtimes E' \xrightarrow{\Delta^*} \mathrm{End}\, E \otimes \omega \xrightarrow{tr} \omega \text{ —— } \omega / d\Omega^{n-1}$$

is an analogue of algebra $\mathcal{A}(E)$ 1.4 (cf. [5 , 2.8]).

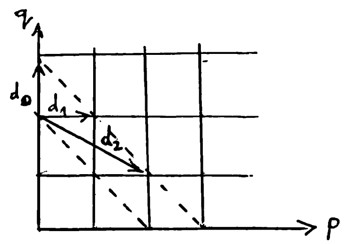
§3. Twisted complexes.

3.1. <u>Twisted bicomplexes</u>. Let $A^{\cdot\cdot} = \{A^{pq}\}_{p,q \in \mathbb{Z}}$ be a bigraded abelian group, $t(A)$ the corresponding simply graded group, i.e. $t(A)^i = \sum_{p+q=i} A^{pq}$. Let $d : t(A) \longrightarrow t(A)$ be an endomorphism of degree +1. It has components $d = \sum d_i$, d_i increases the first degree by i, i.e. $d_i = \sum d_i^{pq}$, $d_i^{pq} : A^{pq} \longrightarrow A^{p+i,\ q-i+1}$.

A pair (A, d) is called <u>a twisted bicomplex</u> if

(a) $d_i = 0$ for $i < 0$, i.e. d respects the filtration by the first degree on A.

(b) $d^2 = 0$.



Assuming the condition a), b) is equivalent to the equalities

(3.1.1) $d_0^2 = 0$, $d_0 d_1 + d_1 d_0 = 0$, $d_0 d_2 + d_1 d_1 + d_2 d_0 = 0, \ldots$

$$\sum_{i=0}^{r} d_i d_{r-i} = 0, \ \ldots$$

Thus, if $d_i = 0$ for $i > 1$, we get a bicomplex (with anticommuting differentials d_0, d_1).

Let us denote $H_I(A)$ the cohomology of A with respect to d_0. From the third equation of (3.1.1) follows that $d_1 d_1$ induces zero on $H_I(A)$. We have a spectral sequence

(3.1.2) $$E_2^{pq} = H_{II}^p(H_I^q(A)) \Longrightarrow H^{p+q}(t(A))$$

were H_{II} denotes cohomology of d_1 on H_I.

<u>Remark</u>. In practice (cf. for example 3.2.5.1) one often meets a system of differentials d_i satisfying the equations

(3.1.1)' $$\sum_{i=0}^{r} (-1)^i d_i d_{r-i} = 0$$

By modifying d_i : $d_i^{pq} = (-1)^p d_i^{pq}$ we get differentials satisfying (3.1.1).

3.2. Twisted group actions.

In this n° we generalize a notion of a group action on a complex

3.2.1. Let $M^\bullet$ be a complex of abelian groups, $End^\bullet(M^\bullet) = Hom^\bullet(M^\bullet, M^\bullet)$ a complex of endomorphisms of $M^\bullet$, i.e.

$$End^i(M^\bullet) = Hom_{\text{graded groups}}(M^\bullet, M^\bullet[i]),$$

for $f \in End^i(M^\bullet)$

(3.2.1.1) $D(f) := [d_M, f] = df + (-1)^{i-1} fd.$

(3.2.1.2) NB! Agreement. In the following we assume that $End^\bullet M$ acts on $M^\bullet$ from the right; in particular (3.2.1.1) means in usual notations

$$D(f)(x) = f(dx) + (-1)^{i-1} d(f(x)).$$

3.2.2. Definition. Let G be a group. A twisted G-action on $M^\bullet$ is a sequence of maps

$$h_i : G^i \longrightarrow End^{1-i}(M^\bullet), \quad i \geq 1,$$

satisfying equations

$$h_1(e) = Id_{M^\bullet} ;$$

(3.2.2.1) $$Dh_i(g_1,\ldots,g_i) = \sum_{j=1}^{i-1} (-1)^j (h_{i-1}(g_1,\ldots,g_j g_{j+1},\ldots,g_i) - h_j(g_1,\ldots,g_j) h_{i-j}(g_{j+1},\ldots,g_i))$$

We call $M^\bullet$ a twisted G-complex.

Thus we have

$$Dh_1 = 0; \quad Dh_2(g_1, g_2) = h_1(g_1)\, h_1(g_2) - h_1(g_1 g_2), \quad \text{etc.}$$

In other words, we have maps of complexes $h_1(g) : M^\bullet \longrightarrow M^\bullet$; homotopies connecting $h_1(g_1)h_1(g_2)$ with $h_1(g_1g_2)$ and so on.

3.2.2.2. In particular, cohomology groups $H^i(M^\bullet)$ have a usual right G-action.

3.2.3. Example. Let R be a commutative ring, M a right $R[G]$-module. Choose a projective resolution

$$P^{\cdot} : \ldots \longrightarrow P^{-n} \longrightarrow P^{-n-1} \longrightarrow \ldots \longrightarrow P^{o} \longrightarrow M \longrightarrow 0$$

of M over R. The multiplications $m \longmapsto mg$ may be lifted to maps $h_1(g) : P^{\cdot} \longrightarrow P^{\cdot}$; for every g_1, g_2 $h_1(g_1)h_1(g_2)$ is homotopic to $h_1(g_1g_2)$. More generally, one has

3.2.3.1. Proposition (cf. 3.3.4.1). There is a sequence of maps $h_i : G^i \longrightarrow End^{1-i}(M^{\cdot})$ with $h_1(g)$ as above, defining a twisted G-action on $P^{\cdot}$.

Proof. Suppose we have already h_p for $p \leq i - 1$. One easily checks that

$$D\Big(\sum_{j=1}^{i-1} (-1)^j \big(h_{i-1}(g_1,\ldots, g_ig_{j+1},\ldots, g_i) - \\ - h_j(g_1,\ldots,g_j)h_{i-j}(g_{i+1},\ldots, g_i)\big)\Big) = 0$$

hence, since $H^q(\mathrm{End}\ P^{\cdot}) = H^q(\mathrm{End}\ M) = 0$ for $q < 0$, there exists $h_i(g_i,\ldots, g_i)$ satisfying (3.2.2.1). ∎

3.2.4. Remark. In the terminology of [8] a twisted G-complex is just a universal pseudo-functor from the category $\underline{G}$ with $\mathrm{Ob}\ \underline{G} = \cdot$, $\mathrm{Mor}\ \underline{G} = G$ to the category of complexes.

3.2.5. Let $M^{\cdot}$ be a twisted G-complex. Define a twisted bicomplex $C^{\cdot}(G, M^{\cdot})$ as follows. Put

$$C^p(G, M^q) = \mathrm{Hom}(G^p, M^q), \quad p \geqslant 0.$$

(Hom as sets!). For $f = f(g_1,\ldots,g_i) \in C^i(G, M^{\cdot})$ put (recall that we write action of d_M from the right, see 3.2.1)

$$(3.2.5.1)\begin{cases} d_of(g_1,\ldots,g_i) = (-1)^i f(g_1,\ldots,g_i)d_M ; \\ d_1f(g_1,\ldots,g_{i+1}) = -f(g_2,\ldots,g_{i+1}) + \sum_{j=1}^{i-1} (-1)^{j-1} f(g_1,\ldots,g_jg_{j+1},\ldots \\ \qquad \ldots,g_i) + (-1)^i f(g_1,\ldots,g_i)\, h_1(g_{i+1}); \\ d_rf(g_1,\ldots,g_{i+r}) = (-1)^i f(g_1,\ldots,g_i)h_r(g_{i+1},\ldots,g_{i-r}), \quad r > 1. \end{cases}$$

(3.2.5.2). Lemma-definition. With the above d_r $C^{\cdot}(G, M^{\cdot})$ is a twisted bicomplex, i.e. putting $d = \sum d_r$ we have $d^2 = 0$.

Proof (cf. [3]). For $f = \sum_{i \geq 0} f_i \in \sum \mathcal{H}om(G^i, M^{\cdot})$,

$h = \sum_{i \geq 1} h_i \in \sum_i \mathcal{H}om(G^i, End^{1-i}(M^{\cdot}))$ put $h_0 = d_{M^{\cdot}} \in End^1(M^{\cdot})$;

$$f\delta(g_1,\dots,g_{i+1}) = -f_i(g_2,\dots,g_{i+1}) + \sum_{j=1}^{i} (-1)^{j-1} f_i(g_1,\dots,g_j g_{j+1},\dots,g_{i+1}),$$

$$h\hat{\delta}(g_1,\dots,g_i) = \sum_{j=1}^{i-1} (-1)^{j-1} h_{i-1}(g_1,\dots,g_i g_{j+1},\dots,g_i);$$

$$f\cdot h(g_1,\dots,g_i) = \sum_{j=0}^{i} (-1)^j f_j(g_1,\dots,g_j)\, h_{i-j}(g_{j+1},\dots,g_i);$$

$$h\cdot h(g_1,\dots,g_i) = \sum_{j=0}^{i} (-1)^j h_j(g_1,\dots,g_j)\, h_{i-j}(g_{j+1},\dots,g_i).$$

In these notations (3.2.2.1) takes the form of Maurer-Cartan equation:

$$h\hat{\delta} + h\cdot h = 0,$$

and (3.2.5.1) -

$$d(f) = f\delta + fh.$$

On the other hand one easily checks that

$$\delta^2 = 0; \qquad \delta h + h\delta = h\hat{\delta},$$

hence

$$d^2 = (\delta + h)^2 = h\hat{\delta} + h^2 = 0 \quad \blacksquare$$

Example. When we have a usual action of G on $M^{\cdot}$, i.e. $h_i = 0$ for $i \geq 2$, then $C^{\cdot}(G, M^{\cdot})$ is (up to signs) the ordinary complex of cochains of G with coefficients in $M^{\cdot}$.

Cohomology groups $H^i(C^{\cdot}(G, M^{\cdot}))$ we'll denote $H^i(G, M^{\cdot})$.

A spectral sequence (3.1.2) for $C^{\cdot}(G, M^{\cdot})$ takes the form

(3.2.5.3) $\quad H^p(G, H^q(M^{\cdot})) \Longrightarrow H^{p+q}(G, M^{\cdot})$

3.2.6. Twisted extensions. Let M, N be G-modules. A twisted n-fold G-extension of M by N is a twisted G-complex of the form

$$0 \longrightarrow N \longrightarrow M_n \longrightarrow \dots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

which is exact as a complex of groups and such that all components of homotopies h_i, $i \geq 2$, going from M or into N, are zero.

Every such extension defines coboundary maps

$$\partial : H^i(G, M) \longrightarrow H^{i+n}(G, N)$$

and in fact, an element of $\mathrm{Ext}^n_G(M, N)$.

3.3. <u>Twisted complexes of sheaves</u>.

3.3.1. Let X be a topological space, $\underline{U} = \{U_\alpha\}$ its open covering. <u>A twisted complex of sheaves $F^\cdot$ over U</u> consists of

(a) A complex of abelian sheaves $F^\cdot_\alpha$ over U_α for each α,

(b) For all $p \geq 0$ and p-tuples $(\alpha_0, \ldots, \alpha_p)$ a map of graded sheaves

$$h_{\alpha_0 \ldots \alpha_p} : F^\cdot_{\alpha_0}\big|U_{\alpha_0 \ldots \alpha_p} \longrightarrow F^\cdot_{\alpha_p}\big|U_{\alpha_0 \ldots \alpha_p}[-p+1]$$

(where $U_{\alpha_0 \ldots \alpha_p} := \bigcap_{i=0}^{p} U_{\alpha_i}$),

such that

(c) h_α = differential in $F^\cdot_\alpha$;

$$(3.3.1.1) \qquad Dh_{\alpha_0 \ldots \alpha_p} = \sum_{j=1}^{p} (-1)^j (h_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_p} - h_{\alpha_0 \ldots \alpha_j} h_{\alpha_j \ldots \alpha_p}),$$

where $Dh := dh + (-1)^{\deg h} hd$ and we write the action of h from the right (3.2.1.2), cf. 3.2.2.1.

Thus, $h_{\alpha\beta}$ is a map of complexes $F^\cdot_\alpha\big|U_{\alpha\beta} \longrightarrow F^\cdot_\beta\big|U$, $h_{\alpha\beta\gamma}$ is a homotopy between $h_{\alpha\gamma}$ and $h_{\alpha\beta}h_{\beta\gamma}$, etc. So cohomology sheaves $\mathcal{H}^i(F^\cdot_\alpha)$ glue by means of maps induced by $h_{\alpha\beta}$ in sheaves over X which we denote by $\mathcal{H}^i(F^\cdot)$

3.2.3. <u>Example</u>. If we have a complex of sheaves $F^\cdot$ over X, then putting $F^\cdot_\alpha = F^\cdot|_{U_\alpha}$, $h_{\alpha\beta}$ be canonical isomorphisms and $h_{\alpha_0 \ldots \alpha_p} = 0$ for $p > 1$, we get a twisted complex.

3.3.3. Let $F^\cdot$ be a twisted complex of sheaves. Put

$$C^p(U, F^q) = \sum_{\alpha_0, \ldots, \alpha_p} \Gamma(U_{\alpha_0 \ldots \alpha_p}, F^q_{\alpha_p})$$

Let us introduce on $C^\cdot(U, F^\cdot) := \sum_{p,q} C^p(U, F^q)$ a structure of a twisted bicomplex. Namely, for $f = (f_{\alpha_0 \dots \alpha_p}) \in C^p(U, F^q)$ put

$$(3.3.3.1) \quad \begin{cases} (d_0 f)_{\alpha_0 \dots \alpha_p} = (-1)^p f_{\alpha_0 \dots \alpha_p} d_{F^\cdot_{\alpha_p}} \\ (d_1 f)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{j=0}^{p} (-1)^{j-1} f_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_{p+1}} + \\ \qquad\qquad + (-1)^p f_{\alpha_0 \dots \alpha_p} h_{\alpha_p \alpha_{p+1}} ; \\ (d_r f)_{\alpha_0 \dots \alpha_{p+r}} = (-1)^p f_{\alpha_0 \dots \alpha_p} h_{\alpha_p \dots \alpha_{p+r}} \quad \text{for } r > 1, \end{cases}$$

cf. (3.2.5.1). As in (3.2.5.2) one verifies that we get a twisted bicomplex. Its cohomology groups we'll denote $H^\cdot(\underline{U}, F^\cdot)$.

A spectral sequence (3.1.2) takes the form

$$(3.3.3.2) \qquad \check{H}^p(\underline{U}, \mathcal{H}^q(F^\cdot)) \Longrightarrow H^{p+q}(\underline{U}, F^\cdot)$$

where in the l.h.s. stands usual Cech cohomology.

If $F^\cdot$ arises from a complex of sheaves over X, 3.3.2, then $H^\cdot(\underline{U}, F^\cdot)$ is Čech cohomology.

3.3.4. Example. Twisted resolutions.

Let F be a sheaf over X. A twisted complex $E^\cdot$ over U is called a twisted resolution of F if $\mathcal{H}^0(E^\cdot) = F$, $\mathcal{H}^i(E^\cdot) = 0$ for $i \neq 0$.

For example, let X be a scheme and F be a sheaf of $\mathcal{O}_X$-modules. Choose over sufficiently small open covering $\underline{U} = \{U_\alpha\}$ a left locally free resolutions $E^\cdot_\alpha \longrightarrow F|_{U_\alpha}$.

3.3.4.1. Proposition (cf. [2, 2.4], [3, 1.3], [7, 3.2.9]). There exist maps $h_{\alpha_0 \dots \alpha_p} : E^\cdot_{\alpha_0}|U_{\alpha_0 \dots \alpha_p} \longrightarrow E^\cdot_{\alpha_p}|U_{\alpha_0 \dots \alpha_p}[-p+1]$ defining on $E^\cdot$ a structure of twisted resolution of F.

Proof. The same as for 3.2.3.1. ∎

3.3.4.2. For two twisted complexes of sheaves $F^\cdot_1$, $F^\cdot_2$ over $\underline{U}$ call a (naive) map $f : F^\cdot_1 \longrightarrow F^\cdot_2$ a family of maps of complexes $f : F^\cdot_{1,\alpha} \longrightarrow F^\cdot_{2,\alpha}$ such that $f_{\alpha_0} h_{\alpha_0 \dots \alpha_p} = h_{\alpha_0 \dots \alpha_p} f_{\alpha_p}$ for all $(\alpha_0, \dots, \alpha_p)$.

Call f quasiisomorphism if all f_α are quasiisomorphisms.

Then a twisted resolution of a sheaf E is the same as quasiisomosphism $f : E^\cdot \longrightarrow \underline{F}$ from some twisted complex to the trivial twisted complex associated with F (concentrated in degree zero), 3.3.2.

3.3.5. Ext's. Let $E^\cdot$ be a twisted complex over $\underline{U}$ and F be sheaf over X. Define a twisted bicomplex $Hom(E^\cdot, F)$ as follows. Put

$$Hom(E^\cdot, F)^{pq} = \sum_{\alpha_o, \dots, \alpha_p} Hom(E^{-q}_{\alpha_o}|_{U_{\alpha_o \dots \alpha_p}}, F|_{U_{\alpha_o \dots \alpha_p}})$$

For $f = (f_{\alpha_o \dots \alpha_p}) \in Hom(E^\cdot, F)^{pq}$ put

$$(3.3.5.1)\quad \begin{cases} (d_o f)_{\alpha_o \dots \alpha_p} = (-1)^p \, df_{\alpha_o \dots \alpha_p} \\ (d_1 f)_{\alpha_o \dots \alpha_{p+1}} = (-1)^p \Big[\check{h}_{\alpha_o \alpha_1} \cdot f_{\alpha_1 \dots \alpha_{p+1}} + \\ \qquad + \sum_{j=1}^{p+1} (-1)^j f_{\alpha_o \dots \hat{\alpha}_j \dots \alpha_p} \Big] \\ (d_r f)_{\alpha_o \dots \alpha_{p+r}} = (-1)^p \, \check{h}_{\alpha_o \dots \alpha_r} f_{\alpha_r \dots \alpha_{p+r}} , \end{cases}$$

cf. 3.3.3.1, where $\check{h}_{\alpha_o \dots \alpha_r} : Hom(E^\cdot_{\alpha_r}|_{U_{\alpha_o \dots \alpha_r}}, F) \longrightarrow$ $\longrightarrow Hom(E^\cdot_{\alpha_o}|_{U_{\alpha_o \dots \alpha_r}}, F)$ is induced by $h_{\alpha_o \dots \alpha_r}$, and we write h to the left.

If X is a scheme, $\underline{U}$ is an affine open covering (or over $\mathbb{C}$ a covering by Stein open sets), F, G coherent $\mathcal{O}_X$-modules, $E^\cdot \longrightarrow F$ a twisted locally free resolution 3.3.4.1 then $H^\cdot\, Hom(E^\cdot, G) = Ext^\cdot(F,G)$, and a spectral sequence (3.1.2) associated with $Hom(E^\cdot, G)$ is a usual spectral sequence from local to global Ext's:

$$(3.3.5.2)\qquad H^p(\underline{U}, \underline{Ext}^q(F, G)) \Longrightarrow Ext^{p+q}(F, G)$$

3.3.6. Remark. Of course, all definitions and results of 3.3 extend in the evident way to sheaves over arbitrary Grothendieck topology, and the contents of 3.2 corresponds to the case of the topos of G-sets.

In the next section we'll need a site whose objects are open domains $U \subset \mathbb{C}^n$ and maps - open holomorphic monomorphisms.

§4. Local calculations

4.1. Koszul resolution of the diagonal. Let $R = \mathbb{C}[[x^1,\dots,x^n; y^1,\dots,y^n]]$ be the ring of formal power series, $V = \sum_{i=1}^{n} Re^i$ a free R-module; its elements will be denoted $f = (f_1,\dots,f_n) = f_i e^i$, $f_i \in R$; $\bar{R} = R/(y^1-x^1, \dots, y^n-x^n) \cong \mathbb{C}[[x^1, \dots, x^n]]$. Denote by K. the Koszul resolution of $\bar{R}$ over R:

$$K. : \quad 0 \longrightarrow K_n \xrightarrow{d} K_{n-1} \xrightarrow{d} \dots \longrightarrow K_o \longrightarrow 0$$

where $K_i = \Lambda^i_R V$; differential d is the interior multiplication by $x = x^i e_i \in V^*$, in other words

$$(4.1.1) \quad d(e^{i_1} \wedge \dots \wedge e^{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x^{i} e^{i_1} \wedge \dots \wedge \widehat{e^{i_r}} \wedge \dots \wedge e^{i_p}$$

Homotopy:

For $p \geqslant 0$ let $S_p : R \longrightarrow V$ be the following k-linear map:

$$(4.1.2) \quad S_p(f) = \int_0^1 t^p (\partial_i f)(x, t(y-x))\, dt\, e^i$$

where $\partial_i = \partial/\partial y^i$. Put $s_p : \Lambda^p V \longrightarrow \Lambda^{p+1} V$ to be

$$(4.1.3) \quad s_p(fe^{i_1} \wedge \dots \wedge e^{i_p}) = S_p(f) \wedge e^{i_1} \wedge \dots \wedge e^{i_p}.$$

If we put $K_{-1} = \bar{R}$; $d_o : K_o \longrightarrow K_{-1} : f(x,y) \longmapsto f(x,x)$, $s_{-1} : \mathbb{C}[[x]] \longrightarrow \mathbb{C}[[x, y]]$ - natural inclusion then we have

$$(4.1.4) \quad d_{i+1}s_i + s_{i-1}d_i = id(K_i), \quad i \geqslant -1$$

4.2. Group of local coordinate transformations

4.2.1. Let G be a group whose elements are n-tuples of power series $\varphi(x) = (\varphi^1(x),\dots,\varphi^n(x))$, $\varphi^i(x) \in \mathbb{C}[[x^1,\dots,x^n]]$, such that

$\varphi(0) = 0$ and $\partial\varphi(0) := \| \frac{\partial\varphi^i}{\partial x^j}(0) \| \in GL_n(\mathbb{C})$. We put $\varphi\psi(x) =$
$= \varphi(\psi(x))$.

G acts on R and $\overline{R}$ from the right by the rule

(4.2.1.1) $\quad \overline{f}\cdot\varphi = {}^{\varphi}\overline{f} := \overline{f}(\varphi(x)), \quad \overline{f} \in \overline{R},$

(4.2.1.2) $\quad f\cdot\varphi = {}^{\varphi}f := f(\varphi(x), \varphi(y)), \quad f \in R$

4.2.2. Let Ω^1 be a right G-module whose elements are sums $f\,dx := f_i(x)\,dx^i$, $\; f_i(x) \in \overline{R}$ with G-action

$${}^{\varphi}(f\,dx) = {}^{\varphi}f\,d\varphi(x) = {}^{\varphi}f_i\,\partial_j\varphi^i(x)\,dx^j\;;$$

put $\Omega^i = \Lambda^i_{\overline{R}}\Omega^1$ with the diagonal G-action. Next, put $\Omega^0 = \overline{R}$, $\omega := \Omega^n$. Thus, elements of ω are expressions $f\,dx^1\wedge\ldots\wedge dx^n$ with G-action

$${}^{\varphi}(f\,dx^1\wedge\ldots\wedge dx^n) = {}^{\varphi}f\cdot\det\partial\varphi\,dx^1\wedge\ldots\wedge dx^n.$$

4.2.3. <u>Chern classes</u>. For $1 \le i \le n$ define i-cocycle $ch_i \in Z^i(G, \Omega^i)$ with coefficients in Ω^i by the formula

$$ch_i(\varphi_1,\ldots,\varphi_i) = \frac{1}{i!}\operatorname{tr}\left[\partial(\varphi_1\ldots\varphi_n)^{-1}\,{}^{\varphi_2\ldots\varphi_n}d\varphi_1\wedge{}^{\varphi_3\ldots\varphi_n}d\varphi_2\ldots\wedge d\varphi_n\right]$$

As usually, put

$$td_i = P_i(ch_1,\ldots,ch_i) \in Z^i(G,\Omega^i)$$

where P_i is a polynomial expressing the i-th homogeneous component of the power series

$$F(T_1,\ldots,T_p) = \prod_{p=1}^{i}\frac{T_p}{1-e^{-T_p}}, \qquad \deg T_p = p,$$

through $\frac{1}{p!}\sum_{q=1}^{i} T_q{}^p, \quad 1 \le p \le i.$

For example, one has $td_1 = \frac{1}{2}ch_1$;

(4.2.3.2) $\quad td_2 = \frac{1}{8}ch_1^2 - \frac{1}{12}ch_2$

4.3. Twisted G-action. Now let us extend the G-action on R (4.2.1.1) to a pseudo-G-action on K.

First, define operators $h(\varphi) : K. \longrightarrow K.$, $\varphi \in G$. On K_o put $h(\varphi)_o : R \longrightarrow R$ to be $f \longmapsto {}^{\varphi}f$ (4.2.1.2). Let us look for $h(\varphi)_1 :$ $: V \longrightarrow V$ in the form $f = (f_i) \longmapsto {}^{\varphi}f \cdot A(\varphi)$ where $A(\varphi) : V \longrightarrow V$ is R-linear operator. The condition $h(\varphi)_1 d = dh(\varphi)_o$ is equivalent to

$$A(\varphi)\cdot(y - x) = \varphi(y) - \varphi(x) \tag{4.3.1}$$

i.e. $a_i^j(\varphi)\cdot(y^i - x^i) = \varphi^j(y) - \varphi^j(x)$, where $A = \| a_i^j \|$. Moreover, if (4.3.1) is satisfied then if we define $h(\varphi)_i : \Lambda^i V \longrightarrow \Lambda^i V$ to be $f \longmapsto {}^{\varphi}f \cdot \Lambda^i A(\varphi)$, where $\Lambda^i A(\varphi) : \Lambda^i V \longrightarrow \Lambda^i V$ is a R-linear operator induced by A, then so defined $h(\varphi) : K. \longrightarrow K.$ is a morphism of complexes.

For $n > 1$ (4.3.1) has a large set of solutions. But we choose a distinguished one:

$$A(\varphi) = \int_0^1 dt\, \partial\varphi(y + t(y - x)) \tag{4.3.2}$$

cf. (4.1.2), where $\partial\varphi := \| \partial_i \varphi^j \|$, $\partial_i := \partial/\partial x^i$.

4.3.3. Problem. Find R-linear operators

$$A_i(\varphi_1, \dots, \varphi_i) : K. \longrightarrow K.[-i + 1]$$

where $K.[i]_j = K_{j-i}$ such that

$$\begin{aligned} &{}^{\varphi_1 \cdot \ldots \cdot \varphi_i} d \cdot A(\varphi_1, \dots, \varphi_i) + (-1)^i A_i(\varphi_1, \dots, \varphi_i) d = \\ &= \sum_{j=1}^{i-1} (-1)^j (A_{i-1}(\varphi_1, \dots, \varphi_j \varphi_{j+1}, \dots, \varphi_i) - \\ &- {}^{\varphi_{j+1} \cdot \ldots \cdot \varphi_i} A_j(\varphi_1, \dots, \varphi_i)\, A(\varphi_{j+1}, \dots, \varphi_i)) \end{aligned} \tag{4.3.3.1}$$

where ${}^{\varphi}d := \lrcorner\, (\varphi^i(y) - \varphi^i(x))e_i$ (recall that we write operators to the right, cf. 3.2.1); $A_1(\varphi) = \oplus\, \Lambda^i A(\varphi)$; and $A_i(\varphi_1, \dots, \varphi_i)_o : \Lambda^o V \longrightarrow \Lambda^{i-1} V$ is zero for $i > 1$.

Having such A_i we can define $h_i(\varphi_1,\ldots,\varphi_i)$ by

$$h_i(\varphi_1,\ldots,\varphi_i)f = {}^{\varphi_1\cdot\ldots\cdot\varphi_i}f\cdot A_i(\varphi_1,\ldots,\varphi_i)$$

and (4.3.3.1) is equivalent to (3.2.2.1).

Since K. is acyclic, such A_i exist (cf. 3.2.3); moreover, using homotopy s (4.1.3) one can easily write expressions for A_i from A_1 as in [1].

4.4. The case n = 2.

From now up to 4.7 suppose that n = 2. So Koszul complex has the form

$$0 \longrightarrow \Lambda^2 V \xrightarrow{d} V \longrightarrow R \longrightarrow 0$$

The only nontrivial $A_2(\varphi,\psi)_1 : V \longrightarrow \Lambda^2 V$ is uniquely determined by condition

$$d\cdot A_2(\varphi,\psi)_1 = {}^{\psi}A(\varphi)\,A(\psi) - A(\varphi\psi).$$

4.4.1. Theorem. Put $B(\varphi,\psi) = {}^{\psi}A(\varphi)\,A(\psi) - A(\varphi\psi)$. Then

$$B(\varphi,\psi) = -H(\varphi,\psi)\begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix}(-(y^2 - x^2),\ y^1 - x^1) + O(y - x)^3$$

where $H = H(x) \in \mathfrak{gl}_2(R)$ is defined by

$$(4.4.1.1)\quad H(\varphi,\psi)\,dx^1\wedge dx^2 = \frac{1}{12}\,{}^{\psi}d(\partial\varphi)\wedge d(\partial\psi),$$

i.e.

$$H(\varphi,\psi) = \frac{1}{12}\left[\partial_1({}^{\psi}\partial\varphi)\,\partial_2(\partial\psi) - \partial_2({}^{\psi}\partial\varphi)\,\partial_1(\partial\psi)\right]$$

Proof. Direct calculation. ∎

4.5. Dual Koszul complex.

4.5.1. This is by definition the complex $K^\cdot = \mathrm{Hom}(K., R)$ with (pseudo)-G-action induced by the above action on K. and standard action (4.2.1.2) on R.

Explicitly:

$$K^\cdot :\quad \underset{\substack{\| \\ R}}{K^0} \xrightarrow{d^0} \underset{\substack{\| \\ R^2}}{K^1} \xrightarrow{d^1} \underset{\substack{\| \\ R}}{K^2}$$

$$d^o(f) = \begin{pmatrix} f\cdot(y^1 - x^1) \\ f\cdot(y^2 - x^2) \end{pmatrix}; \quad d^1 \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} = (-(y^2 - x^2),\ y^1 - x^1) \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}.$$

Homotopy. We shall need $s^2 : K^2 \to K^1$ up to the second order:

$$s^2(\alpha_{00}(x) + \alpha_{10}(x)(y^1 - x^1) + \alpha_{01}(x)(y^2 - x^2) + \alpha_{20}(x)(y^1 - x^1)^2 +$$

$$+ 2\alpha_{11}(x)(y^1 - x^1)(y^2 - x^2) + \alpha_{02}(x)(y^2 - x^2)^2 + \dots) =$$

$$= \begin{pmatrix} -\alpha_{01} - \alpha_{11}(y^1 - x^1) - \alpha_{02}(y^2 - x^2) \\ \alpha_{10} + \alpha_{20}(y^1 - x^1) + \alpha_{11}(y^2 - x^2) \end{pmatrix} + O(y - x)^2;$$

and $s^1 : K^1 \to K^o$ up to the first order:

$$s^1\left(\begin{pmatrix} \alpha^1 + \alpha^1_1(y^1 - x^1) + \alpha^1_2(y^2 - x^2) \\ \alpha^2 + \alpha^2_1(y^1 - x^1) + \alpha^2_2(y^2 - x^2) \end{pmatrix} + \dots\right) = \frac{1}{2}(\alpha^1_1 + \alpha^2_2) + O(y - x).$$

4.5.1.1. G-action: for $\varphi \in G$ $fh(\varphi)^o = {}^{\varphi}f$ $(f \in K^o)$; $fh(\varphi)^1 = A(\varphi)^{-1}\,{}^{\varphi}f$ $\left(f = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} \in K^1\right)$; $fh(\varphi)^2 = \det A(\varphi)^{-1}\cdot{}^{\varphi}f$ $(f \in K^2)$.

4.5.2. Lemma. Put $\tilde{B}(\varphi,\psi) = A(\psi)^{-1}\,{}^{\psi}A(\varphi)^{-1} - A(\varphi\psi)^{-1}$. Then

$$\tilde{B}(\varphi,\psi) \equiv -\partial(\varphi\psi)^{-1} B(\varphi,\psi)\, \partial(\varphi\psi)^{-1} \bmod (y - x)^3$$

where $B(\varphi,\psi)$ is as in 4.4.1. ■

4.5.3. Corollary. Homotopy $h_2(\varphi,\psi)^2 : K^2 \to K^1$ such that

$$dh_2(\varphi,\psi)^2 = h(\varphi)\, h(\psi) - h(\varphi\psi)$$

is defined by $h_2(\varphi,\psi)^2(f) =$

$$= (\det \partial(\varphi\psi)^{-1}\cdot\partial(\varphi\psi)^{-1}\cdot H(\varphi,\psi) \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix} + O(y - x)^2)\cdot {}^{\varphi\psi}f,$$

cf. 4.4.1.1.

Proof. This follows from 4.4.1 and the equality

$${}^{\varphi\psi}(-(y^2 - x^2),\ y^1 - x^1) = (-(y^2 - x^2),\ y^1 - x^1)\partial(\varphi\psi)^{-1} + O(y - x)^2. \quad ■$$

4.6. <u>What we have to calculate</u>.

The complex we need is the tensor product $M^{\cdot} = K^{\cdot} \otimes \omega_y$ where ω_y denotes the G-module $\{f(y)\, dy^1 \wedge dy^2\}$, 4.2.2. To obtain formulas of G-action on $M^{\cdot}$ we have to multiply formulas for G-action on $K^{\cdot}$ from the preceding n° on $\det \partial\varphi(y)$ (for $h_1(\varphi)$) and $\det \partial(\varphi\psi)(y)$ (for $h_2(\varphi,\psi)$).

Thus we have an exact sequence

$$(4.6.1) \qquad 0 \longrightarrow M^o \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \overline{M}^2 \longrightarrow 0$$

where $\overline{M}^2 = \Omega^o$, 4.2.2. M^o has a filtration by powers of $(y - x)$ whose first quotient is ω; let

$$(4.6.2) \qquad 0 \longrightarrow \omega \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \Omega^o \longrightarrow 0$$

be the exact sequence induced from (4.6.1) by the projection $M^o \longrightarrow \omega$. It is the twisted G-extension of Ω^o by ω, 3.2.6, and we have to calculate its class, i.e. $\delta(1) \in Z^2(G,\omega)$, $1 \in \Omega^{oG}$.

Let us draw a part of $C^{\cdot}(G, M^{\cdot})$ we need:

$$\begin{array}{ccccc}
M^2 & \xrightarrow{\ d_1\ } & \mathrm{Hom}(G, M^2) & & \\
 & \searrow^{d_2} & \uparrow{\scriptstyle d_o = -d} & & \\
 & & \mathrm{Hom}(G, M^1) & \xrightarrow{\ d_1\ } & \mathrm{Hom}(G^2, M^1) \\
 & & & & \uparrow{\scriptstyle d_o = d} \\
 & & & & \mathrm{Hom}(G^2, M^o)
\end{array}$$

Put $m := 1 \in M^2$ - a lifting of $1 \in \Omega^o$. We have to find elements $m_1(\varphi) \in \mathrm{Hom}(G, M^1)$ such that $d_o m_1 = -d_1 m$, i.e.

$$dm_1(\varphi) = -m + mh_1(\varphi)$$

and $m_2(\varphi,\psi) \in \mathrm{Hom}(G^2, M^o)$ such that $d_o m_2 = -d_1 m_1 - d_2 m$, i.e.

$$dm_2(\varphi,\psi) = m_1(\psi) - m_1(\varphi\psi) + m_1(\varphi)h_1(\psi) - mh_2(\varphi,\psi)$$

(here d denotes the differential in $M^{\cdot}$).

Denote by $p : \mathrm{Hom}(G^2, M^o) \longrightarrow \mathrm{Hom}(G^2, \omega)$ the projection. Then

$p(m_2)$ will be the desired cocycle.

Let s denote the homotopy for d_o induced by homotopy of $K^\bullet$, 4.5.1. We put

$$m_1 = -sd_1 m;$$

(4.6.3) $$m_2 = -sd_1 m_1 - sd_2 m = ((sd_1)^2 - sd_2)m$$

4.7. Theorem. We have

(i) $p(sd_1)^2 m = \frac{1}{8}\, ch_1^2$

(ii) $psd_2 m = \frac{1}{12}\, ch_2$

Hence,

$$p(m_2) = td_2$$

cf. 4.2.3.

Proof. (ii) follows immediately from 4.5.3. Let us calculate $(sd_1)^2 m$. We have

$$mh(\varphi) = \det(\partial\varphi(y))\, A(\varphi)^{-1} = 1 + \tfrac{1}{2}\, \mathrm{tr}(\partial\varphi^{-1}(x)\ \partial(\partial\varphi(x)))(y - x) +$$
$$+ C^{20}(\varphi)(y^1 - x^1)^2 + 2\, C^{11}(\varphi)(y^1 - x^1)(y^2 - x^2) + C^{02}(\varphi)(y^2 - x^2)^2 +$$
$$+ O(y - x)^3$$

where we put $\partial = (\partial_1, \partial_2)$; $(y - x) = \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix}$, $\quad C^{ij}(\varphi) \in \overline{R}$.

Hence

$$m_1(\varphi) = -sd_1 m = \begin{pmatrix} -\frac{1}{2}\, \mathrm{tr}(\partial\varphi^{-1}\ \partial_2(\partial\varphi)) - C^{11}(y^1 - x^1) - C^{02}(y^2 - x^2) \\ \frac{1}{2}\, \mathrm{tr}(\partial\varphi^{-1}\ \partial_1(\partial\varphi)) + C^{20}(y^1 - x^1) + C^{11}(y^2 - x^2) \end{pmatrix} + O(y - x)^2$$

(we omit for brevity argument x).

4.7.1. Lemma.

$$\det(\partial\psi)\cdot \partial\psi^{-1}\ {}^{\psi}\begin{pmatrix} -\mathrm{tr}(\partial\varphi^{-1}\ \partial_2(\partial\varphi)) \\ +\mathrm{tr}(\partial\varphi^{-1}\ \partial_1(\partial\varphi)) \end{pmatrix} = \begin{pmatrix} -\mathrm{tr}({}^{\psi}\partial\varphi^{-1}\ \partial_2({}^{\psi}\partial\varphi)) \\ \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\ \partial_1({}^{\psi}\partial\varphi)) \end{pmatrix} \quad \blacksquare$$

From this lemma and the equality $\partial(\varphi\psi) = {}^{\psi}\partial\varphi \cdot \partial\psi$ follows that

$$m_1(\psi) - m_1(\varphi\psi) + m_1(\varphi)h_1(\psi) =$$

$$= \left[C(\psi) - C(\varphi\psi) + \det \partial\psi \cdot \partial\psi^{-1} \cdot {}^{\psi}C(\varphi)\partial\psi + \tfrac{1}{4}E + \tfrac{1}{2}\widetilde{E} \right] \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix} + O(y-x)^2$$

where we put

$$C(\varphi) = \begin{pmatrix} -c^{11}(\varphi) & -c^{02}(\varphi) \\ c^{20}(\varphi) & c^{11}(\varphi) \end{pmatrix} \in \mathfrak{gl}_2(\bar{R}) \ ;$$

and $E, \widetilde{E} \in \mathfrak{gl}_2(R)$ are defined by the equalities

$$\partial\psi \cdot E \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix} = \partial_1(\partial\psi) \begin{pmatrix} \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_2({}^{\psi}\partial\varphi))(y^1 - x^1) \\ -\mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_1({}^{\psi}\partial\varphi))(y^1 - x^1) \end{pmatrix} +$$

$$+ \partial_2(\partial\psi) \begin{pmatrix} \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_2({}^{\psi}\partial\varphi))(y^2 - x^2) \\ -\mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_1({}^{\psi}\partial\varphi))(y^2 - x^2) \end{pmatrix} ;$$

$$\widetilde{E} \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix} = \mathrm{tr}(\partial\psi^{-1}\partial_1(\partial\psi)) \begin{pmatrix} -\mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_2({}^{\psi}\partial\varphi)) \\ \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_1({}^{\psi}\partial\varphi)) \end{pmatrix} (y^1 - x^1) +$$

$$+ \mathrm{tr}(\partial\psi^{-1}\partial_2(\partial\psi)) \begin{pmatrix} -\mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_2({}^{\psi}\partial\varphi)) \\ \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_1({}^{\psi}\partial\varphi)) \end{pmatrix} (y^2 - x^2)$$

Now recall that

$$c_1(\varphi,\psi)^2 = \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_1(\partial\varphi))\mathrm{tr}(\partial\psi^{-1}\partial_2(\partial\psi)) - (1 \longleftrightarrow 2),$$

so $\mathrm{tr}\,\widetilde{E} = c_1(\varphi,\psi)^2$. On the other hand, we have

(4.7.2) Lemma. $\mathrm{tr}\, E = -c_1(\varphi,\psi)^2$.

Proof. If A_1, A_2 are 2×2-matrices such that the second column of A_1 is equal to the first column of A_2, and E is defined by the equality

$$E \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = A_1 \begin{pmatrix} c^2 x^1 \\ -c_1 x^1 \end{pmatrix} + A_2 \begin{pmatrix} c_2 x^2 \\ -c_1 x^2 \end{pmatrix}$$

then $\mathrm{tr}\, E = \mathrm{tr}\, A_1 \cdot c_2 - \mathrm{tr}\, A_2 \cdot c_1$. Applying this to $A_i = \partial\psi^{-1}\partial_i(\partial\psi)$; $c_i = \mathrm{tr}({}^{\psi}\partial\varphi^{-1}\partial_i({}^{\psi}\partial\varphi))$ we obtain the lemma. ∎

Finally, $\mathrm{tr}\, C = 0$, and applying formula for homotopy s^1, 4.5.1, we obtain

$$m_2(\varphi,\psi) = \frac{1}{2}\,tr\left(\frac{1}{2}\widetilde{E} + \frac{1}{4}E\right) + O(y-x) = \frac{1}{8}c_1(\varphi,\psi)^2 + O(y-x)$$

which proves (i). ■

4.8. Let now n be arbitrary. The corresponding complex $M^{\cdot} = K^{\cdot} \otimes \omega_y$ looks like

$$0 \longrightarrow M^o \longrightarrow M^1 \longrightarrow \dots \longrightarrow M^n \longrightarrow \Omega^o \longrightarrow 0$$

Now define $m_i \in Hom(G^i, M^{n-i})$ as in the case $n = 2$, starting from $1 \in \Omega^o$. Then one easily sees that

$$m_n = \sum_{q=1}^{n} m_n^{(q)}$$

where

$$m_n^{(q)} = (-1)^q \sum (sd_{\sigma(1)})^{i_1} (sd_{\sigma(2)})^{i_2} \dots (sd_{\sigma(q)})^{i_q} m$$

the summation being taken over all pairs (inclusion $\sigma : \{1, 2,\dots,q\} \longrightarrow \{1, 2, \dots, n\}$; $(i_1,\dots,i_n)$) such that $\sum_p i_p i(p) = n$.

On the other hand, let

$$td_n = P_n(ch_1,\dots,ch_n) = \sum_{q=1}^{n} P_n^{(q)}(ch_1,\dots,ch_n)$$

where $P_n^{(q)}$ is a part of P_n which contains the sum of products of q factors.

4.8.1. Conjecture. Let $p : Hom(G^n, M^o) \longrightarrow Hom(G^n,\omega)$ be the projection. Then one has

$$p(m_n^q) = P_n^{(q)}(ch_1,\dots, ch_n).$$

For the case $n = 2$ it is just Thm 4.7; $n = 1$ is trivial and is contained in §1.

4.9. Riemann-Roch for surfaces.

Let G denote the category whose objects are open domains $U \subset \mathbb{C}^n$ and morphism - open holomorphic monomorphisms $\varphi: U \longrightarrow V$.

Over G we have a sheaves R, $\overline{R}$ with $R(U)$ = holomorphic functions on $U \times U$, (resp., $\overline{R}(U)$ = holomorphic functions on U); Ω^i, etc.

All the results of nn 4.1-4.9 extend word by word to this situation (cf. 3.3.6).

Now let X be a smooth compact n-dimensional complex variety. Choose an open covering $X = \bigcup U_i$ together with isomorphisms of U_i with open domains in $\mathbb{C}^n$. With these data the above constructions give us a twisted locally free $\mathcal{O}_{X\times X}$ - resolution of the diagonal 3.3.4, and 4.7 just calculates the class $(1_{\mathcal{O}_X}) \in \check{C}(\underline{U},\omega)$ for n = 2. So we get

Theorem. If X is a smooth compact complex surface, then

$$\chi(X) = \int_X td(\mathcal{T}_X)_2 . \quad \blacksquare$$

4.10. We leave to the reader the extension of the previous calculations to the case of surfaces with a bundle. Hint only that one has to use instead of category G from 4.9 a category $G \ltimes GL_m$ with the same objects as in G, and morphisms - pairs (φ, ψ) where $\varphi: U \to V$ as in G and ψ a holomorphic map $U \to GL_m(C)$, cf. [1], §6.

§5. Remarks on constructing of $\mathcal{P}(E)$ in higher dimensions

5.1. Let R, G be as in 4.1 with n = 2. Consider the following complex

$$K. = (K_2 \to K_1 \to K_o) = (R^2 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R)$$

where

$$d_1(f_1, f_2, f_3) = (f_1, f_2, f_3)\cdot \begin{pmatrix} (y^1 - x^1)^2 \\ (y^1 - x^1)(y^2 - x^2) \\ (y^2 - x^2)^2 \end{pmatrix} ,$$

$$d_2(f_1, f_2) = (f_1, f_2) \begin{pmatrix} y^2 - x^2 & -(y^1 - x^1) & 0 \\ 0 & -(y^2 - x^2) & y^1 - x^1 \end{pmatrix}$$

K. is a free R-resolution of $R/(x - y)^2R$. Introduce a (twisted) G-action on K. by formulas: for $\varphi \in G$:

$$fh(\varphi)_o = {}^{\varphi}f \ (f \in K_o);$$
$$fh(\varphi)_1 = {}^{\varphi}f \cdot S^2A(\varphi);$$
$$fh(\varphi)_2 = {}^{\varphi}f \cdot \tilde{A}(\varphi) \cdot \det A(\varphi),$$

where $A(\varphi) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is as in (4.3.2),

$$S^2 A(\varphi) := \begin{pmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22}+a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{pmatrix},$$

$$\tilde{A}(\varphi) := \begin{pmatrix} a_{11} & -a_{21} \\ -a_{12} & a_{22} \end{pmatrix}$$

$h(\varphi,\psi)_2 : K_1 \to K_2$ is defined uniquely by previous formulas.

5.2. One easily sees that $\mathrm{Hom}(K.,\omega)$ defines a canonical twisted G-extension of $\mathcal{D}(\mathcal{O})^{\leq 1}$ by R (i.e. by $\mathcal{O} \boxtimes \mathcal{O}'$).

By adding a gauge group, we get on an arbitrary surface,with vector bundle E, a canonical twisted extension (of length 2) of $\mathcal{D}(E)^{\leq 1}$ by $E \boxtimes E'$, i.e. an analogue of $\mathcal{P}(E)^{\leq 1}$ (cf. 2.4.4).

It would be very interesting to extend the calculations of §4 to this case and prove a cancellation of anomalies conjecture of [5 , 3.4] and more generally, Grothendieck-Riemann-Roch (cf. [5], Appendix) for the families of surfaces.

5.3. It seems undoubtedly that this generalized Koszul construction gives a canonical twisted n-fold extension of $\mathcal{D}(E)^{\leq a}$ by $E \boxtimes E'$ in dimension n for every finite a.

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