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# A PROBLEM ABOUT FEYNMAN INTEGRALS ON RIEMANNIAN MANIFOLDS

Hugo H. Torriani

Abstract: In this expository paper we draw attention to a problem arising in the quantization of a free particle in the presence of curvature. This note is also a minisurvey of some recent developments in the vast areas it touches upon.

The classical hamiltonian of a free particle (i.e., the kinetic energy of the particle) on a riemannian manifold of dimension $n$ and metric $g$ is given by

$$H(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k ,$$

where $(x,\xi) = (x^1,\dots,x^n,\xi_1,\dots,\xi_n)$ are canonical coordinates on the cotangent bundle. Various procedures of quantization lead to a quantum hamiltonian

$$\hat{H} = \frac{\hbar^2}{2}(\Delta + cR) \tag{1}$$

corresponding to $H$, where $\hbar = \frac{h}{2\pi}$ ($h$ being Planck's constant), $\Delta$ is the Laplace-Beltrami operator, $R$ is the scalar curvature associated to $g$, and $c$ is an "author-dependent" constant. (Cf. [19].) For example, $c$ has been found to be equal to $\frac{1}{6}$ (cf. [9], [1]), to $\frac{1}{3}$ (cf. [6]), and to $\frac{1}{4}$ (cf. [8]). Remarkably enough, the authors of the last four papers we have cited use the same basic approach to quantization, namely Feynman's path integrals. It is also remarkable that in [1] and [8] the competing constants are recovered by appealing to physical rules, viz. canonical quantization and Schwinger's quantum action principle, respectively.

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The value $c = \frac{1}{6}$ has also been obtained in a way totally independent from Feynman's integrals by J. Elhadad (cf. [10]), who generalized Souriau's approach to quantization by introducing Maslov's index into the Kostant-Sternberg-Blattner pairing integral for the case of the sphere. Also via geometric quantization that value has been recovered by J. Śniatycki. (Cf. [27], pp. 21, 134.) However, no correction term involving the scalar curvature appears in [25], pp. 79-85.

About a year ago, R. Kuwabara (cf. [19]) showed that from a spectral viewpoint the value $c = \frac{1}{6}$ has a special meaning. We recall (cf. e.g. [4], [3], [2]) that a Zoll manifold is a riemannian manifold all of whose geodesics are closed and have common length $2\pi$. On the sphere $S^n$ there is a smooth family of metrics $g(s)$ $(0 \leq s \leq T)$ such that $(S^n, g(s))$ is a Zoll manifold for each $s$ and $g(0) = g_0$ is the standard metric. For each such $s$ let

$$H_s(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(s;x)\xi_j\xi_k$$

be the classical hamiltonian of a free particle on $(S^n, g(s))$, $\Delta_s$ the Laplace-Beltrami operator on this manifold, $V_s$ a $C^\infty$ function on it such that $V_0$ is a constant, and $\hat{H}_s$ the operator defined by

$$\hat{H}_s = \frac{\hbar^2}{2}(\Delta_s + V_s).$$

Let $\{\lambda_k\}_{k=0}^{\infty}$ be the spectrum of $\hat{H}_0$. By a theorem of A. Weinstein ([29]; cf. also [15]) there exists a constant $M$ (independent of $k$) such that the spectrum of each $\hat{H}_s$ $(0 \leq s \leq T)$ consists of clusters of points $\{\lambda_k^i(s)\}$ $(i = 1,\ldots,N_k;$ $\lambda_k^i(0) = \lambda_k)$ in the intervals $I_k = [\lambda_k - M, \lambda_k + M]$ $(k = 0, 1, 2, \ldots)$. Let us assume that $n \geq 3$ and let us consider the arithmetic mean

$$\bar{\lambda}_k(s) = \frac{1}{N_k}\sum_{i=1}^{N_k} \lambda_k^i(s)$$

of the eigenvalues of $\hat{H}_s$ in the k-th cluster. Then Kuwabara's main result says that $\bar{\lambda}_k(s) \to \lambda_k$ as $k \to \infty$ for any Zoll metric $g(s)$ if and only if $c = \frac{1}{6}$.

The key step in Kuwabara's theorem consists in showing that $\bar{\lambda}_k(s) - \lambda_k$ tends to

$$\frac{\{a_1(0) - a_1(s)\}\hbar^2}{2\,\mathrm{vol}(S^n, g_0)} + O(k^{-2}) \quad \text{as} \quad k \to \infty, \tag{2}$$

where

$$a_1(s) = \int_{S^n}\left(\frac{R_s}{6} - V_s\right)dV_{g(s)}, \tag{3}$$

$R_s$ and $dV_{g(s)}$ being the scalar curvature and the volume element, respectively, associated to the metric $g(s)$. The proof of (2) rests on a formula for the multiplicity of the eigenvalues of $\hat{H}_0$ due to Y. Colin de Verdière (cf. [7]), and on a comparison of two expressions for the asymptotic expansion of the partition function with parameter s. If $V_s = R_s/6$ for each s, then $a_1(s)$ vanishes by (3) and $\overline{\lambda}_k(s) - \lambda_k = O(k^{-2})$ as $k \to \infty$, whence sufficiency follows (with an estimate on the rate of convergence).

If, conversely, $\overline{\lambda}_k(s) \to \lambda_k$ as $k \to \infty$, then $a_1(s) = a_1(0)$ (cf. (2)) for any Zoll metric $g(s)$, and in particular, for any conformal deformation $g(s)$ of $g_0$, i.e., of the form $g_h(s) = f_h(s)g_0$, where $f_h(s)$ is a one-parameter family of positive functions. In case $V_s = cR_s$ for each s, relation (3) yields $a_1(0) = (\frac{1}{6} - c)F(s)$, where

$$F(s) = \int_{S^n} R_s \, dV_{g(s)}$$

is a functional that is shown to be nonconstant by the methods of [18].

Kuwabara's result highlights the following problem, various aspects of which have been considered by several authors since the middle fifties:

PROBLEM. (a) Reconcile the results of DeWitt [9], Cheng [6], Ben-Abraham and Lonke [1], and Dekker [8], within the framework of the Feynman quantization approach.

(b) Furnish a detailed justification for the discrepancy existing in the results of Simms-Woodhouse ([25], pp. 79-85) and Śniatycki ([27], pp. 21, 134), obtained via geometric quantization.

(c) Show that the agreements reached in (a) and (b) match. (Cf. [27], pp. 134 seq., and the references given on p. 23.)

From a mathematical standpoint, interest on this problem would seem to center upon the fact that it lies at the confluence of four large and important areas, namely:

- Measure and Integration, and especially the theory of integration in function spaces;
- Stochastic Processes, and in particular, the subject of random differential equations;
- Differential Geometry, and specifically, spectral problems on riemannian manifolds; and last but not least,
- Quantum Mechanics, and more particularly, quantization processes of classical dynamical systems.

From a physical viewpoint we may mention:

- The obvious necessity of eliminating ambiguities in the Feynman quantization process and of exhibiting its compatibility with geometric quantization in simple cases;
- The fact that the curvature term may be significant in the study of highly

condensed matter states such as white dwarfs, neutron stars and black holes, and also in more normal situations when the curvature is induced by constraints (cf. [1]; [27], p. 120; [22], p. 513); and

- The relationship existing between spectral properties of the laplacian with potentials (cf. [29]) and basic quantum mechanical spectral phenomena, such as the Zeeman effect and the Stark effect. (Cf. e.g. [23], pp. 662 seq.)

It would seem reasonable to begin consideration of our problem by restricting attention to a well-chosen particular class of riemannian manifolds. Of course, the circle would be the simplest nontrivial case (cf. [22], pp. 515-518; [21], p. 27), but its very special nature might perhaps turn out to be a hindrance. Other classes that come to mind are higher-dimensional spheres (cf. [22], pp. 527-531; [21], p. 29; [20], pp. 220-227 and 287-302), spaces of constant curvature (cf. e.g. [30]), unimodular Lie groups (cf. [16], p. 366; the existence of a bi-invariant Haar measure may be useful), and homogeneous spaces (cf. [17], pp. 212-213).

On the other hand, it is clear that a sensible approach to this problem must start with a recasting of the formulations of path integration on manifolds found in the physical literature into a mathematical framework at the level of e.g. [26] or [14]. Probably the very first thing to do would be to look into Wiener processes on riemannian manifolds (cf. [11]) and into the Feynman-Kac type formula (cf. [26], p. 49; [14], p. 47) used in [12]. Incidentally, as this last paper is chiefly concerned with dequantization, the absence of a scalar curvature term in the Schrödinger equation considered therein should not, of course, be construed as being in conflict with the nonzero constants derived in papers that follow the opposite orientation.

Partial results in the general direction of our problem, or side developments motivated by it, would also seem to be worthwhile. For example, it may be of interest to verify whether quasi-classical expansions analogous to those of [12] hold for Schrödinger equations with scalar curvature terms; or to see how the proofs of op. cit. simplify when specialized to one of the important particular classes of riemannian manifolds enumerated above. In addition, it may be fruitful to examine the results of [20] (cf. esp. pp. 280 seq.) in the light of [12]. Another interesting possibility would be to try to find results analogous to Kuwabara's for manifolds other than $(S^n, g(s))$.

Except for a brief mention of heuristic quantization made at the beginning, this note has touched upon some aspects of functional quantization and geometric quantization. It may be surmised however that also the methods of stochastic quantization (cf. e.g. [5], [13], [24]) might throw some light on these questions.

To conclude we return to the first part of the problem formulated above. In view of [9], [1], [10], [27] and [19] it would seem that the value $c = \frac{1}{6}$ for (1) stands better chances to be correct than the values $c = \frac{1}{3}$ of [6] and $c = \frac{1}{4}$ of [8]. (We note in passing the curious fact that [1] is not cited in [19], [8], or in the extensive bibliography of [20]. That article is however quoted in [28], p. 293, and [27], p. 22.) Definitive arguments in favor of just one of these constants would thus appear to be urgently needed. In addition, it would be necessary to furnish convincing explanations as to why the application of Feynman's quantization procedure that led to the other competing constants is flawed —— whether mathematically, or physically, or both.

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