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# REMARKS ON THE SCHOUTEN-NIJENHUIS BRACKET

Peter W. Michor

In 1940 Schouten introduced the differential invariant of two purely contravariant tensor fields. In 1955 Nijenhuis showed that for skew symmetric contravariant tensor fields (also called skew multivector fields) this concomitant satisfies the Jacobi identity and gives a structure of a graded Lie algebra to the space of all multivector fields. The same is true for the symmetric multivector fields.
In 1974 Tulczyjew gave a coordinate free treatment of the bracket for skew multivector fields and clarified its relation to certain differential operators on the space of differential forms, which are similar to those of the better known and more important Frölicher-Nijenhuis bracket for tangent bundle valued differential forms. The Schouten-Nijenhuis bracket can be used to express integrability properties (1.3) and its vanishing is also the condition on a 2-vector field to define a Poisson bracket for functions (a coordinate free proof of this is in 1.4). Recently Koszul explained some relations of this bracket to Lie algebra cohomology. The Schouten-Nijenhuis bracket for symmetric multivector fields is well known to coincide with the Poisson bracket of the associated functions on the cotangent bundle, which are polynomial along the fibres. It will not be treated in this paper - similar results as those treated in this paper are true for it.

In this paper we introduce the Schouten-Nijenhuis bracket for skew multivector fields as extension of the Lie bracket for vector fields satisfying certain properties. We scetch its uses and we rederive the formulas of Tulczyjew concerning the Lie differentials of forms ending up with the definition Tulczyjew started with. Note that the bracket defined here differs in sign from the usual one. In the second part we show that the Schouten-Nijenhuis bracket is natural with respect to f-dependence of multivector fields, and finally, that it is (up to a multiplicative constant) the unique natural concomitant mapping a p-field and a q-field to a p+q-1 - field.

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This paper is in final form and no version of it will appear elsewhere.

## 1. The Schouten-Nijenhuis bracket for skew multivector fields.

1.1. Let $M$ be a smooth manifold, finite dimensional and paracompact. Then the Lie algebra $\mathfrak{X}(M) = \Gamma(TM)$ of vector fields on $M$ is a module over the commutative algebra $C^\infty(M)$ of smooth functions on $M$, and $\mathfrak{X}(M)$ acts on $C^\infty(M)$ as Lie algebra of derivations, via $\theta: \mathfrak{X}(M) \to \mathrm{Der}(C^\infty(M))$. This is sometimes called a Lie-module.
Let us now consider $\Gamma(\Lambda TM)$, the graded commutative algebra of (skew) multivector fields on $M$. It coincides with $\Lambda_{C^\infty(M)}\Gamma(TM)$, the space of skew elements in $\otimes_{C^\infty(M)} \Gamma(TM)$, the $C^\infty(M)$-tensor algebra generated by the $C^\infty(M)$-module $\Gamma(TM)$.

1.2. Theorem: The following bracket is well defined on $\Gamma(\Lambda TM)$ and gives a graded Lie algebra structure with grading $(\Gamma(\Lambda TM), [\;,\;])_p = \Gamma(\Lambda^{p+1}TM)$:

$$[X_1\wedge\ldots\wedge X_p, Y_1\wedge\ldots\wedge Y_q] =$$
$$= \sum (-1)^{p-i+j-1} X_1\wedge\ldots\widehat{X_i}\ldots\wedge X_p\wedge[X_i,Y_j]\wedge Y_1\wedge\ldots\widehat{Y_j}\ldots\wedge Y_q$$
$$= \sum (-1)^{i+j} [X_i,Y_j]\wedge X_1\wedge\ldots\widehat{X_i}\ldots\wedge X_p\wedge Y_1\wedge\ldots\widehat{Y_j}\ldots\wedge Y_q,$$

$[f,U] = \bar{i}(df)U$ for $f$ in $C^\infty(M)$ and $U$ in $\Gamma(\Lambda TM)$.
We also have $[U, V\wedge W] = [U,V]\wedge W + (-1)^{(u-1)v}\, V\wedge[U,W]$, so that $\mathrm{ad}: (\Gamma(\Lambda TM), [\;,\;]) \to \mathrm{Der}(\Gamma(\Lambda TM), \wedge)$ is a homomorphism of graded Lie algebras.

Proof: For vector fields $X_i$ and $Y_j$ and $f$ in $C^\infty(M)$ the following is easily seen to hold: $[X_1\wedge\ldots\wedge X_p, Y_1\wedge\ldots\wedge f.Y_j\wedge\ldots\wedge Y_q] = f.[X_1\wedge\ldots\wedge X_p, Y_1\wedge\ldots\wedge Y_q] + (-1)^{p-1}\,\bar{i}(df)(X_1\wedge\ldots\wedge X_p)\wedge Y_1\wedge\ldots\wedge Y_q$, where $\bar{i}(df)$ is the insertion operator, a derivation of degree $-1$.
The formula given in the theorem defines a priori a bilinear mapping $\Lambda^p\Gamma(TM) \times \Lambda^q\Gamma(TM) \to \Lambda^{p+q-1}\Gamma(TM)$. If we map it into $\Lambda^{p+q-1}_{C^\infty(M)}\Gamma(TM) = \Gamma(\Lambda^{p+q-1}TM)$ then by the formula above and by antisymmetry it factors over $\Gamma(\Lambda^pTM) \times \Gamma(\Lambda^qTM)$. So it is well defined. Then one has to check the graded Jacobi-identity. This is an elementary but very tedious calculation. The last property is rather easily checked. qed.

Remark: The bracket defined in the theorem is the universal extension of the $C^\infty(M)$-Lie module $\mathfrak{X}(M)$ to a graded version.

1.3. Integrability lemma: Let $F \subset TM$ be a 2-dimensional sub vector bundle (a distribution or 2-plane field). Let $U$ in $\Gamma(\Lambda^2TM)$ be a (local) "basis" for it ((so $X_x \in F$ iff $X_x \wedge U = 0$). Then $F$ is integrable if and only if $[U,U] = 0$.

Proof: Let $X$, $Y$ be local vector fields spanning $F$. We may assume that $U = X\wedge Y$. Then $[X\wedge Y, X\wedge Y] = 2\, X\wedge[X,Y]\wedge Y$ by 1.2, which is zero iff $[X,Y]$ is in $F$. qed.

1.4. Characterisation of Poisson structures: Let $P$ be in $\Gamma(\Lambda^2 TM)$. Then the skew symmetric product $\{f,g\} := \langle df\wedge dg, P\rangle$ on $C^\infty(M)$ satisfies the Jacobi identity if and only if $[P, P] = 0$.

Proof: $\{f,g\} = \langle df\wedge dg, P\rangle = \langle dg, \bar{i}(df)P\rangle = \langle 1, \bar{i}(dg)[f,P]\rangle = [g,[f,P]]$.
So $\{f,\{g,h\}\} = [[h,[g,P]],[f,P]]$. Now a straightforward computation involving graded Jacobi identity and skew symmetry of the Schouten-Nijenhuis bracket gives:
$[h,[g,[f,[P,P]]]] = -2(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\})$.
Since $[h,[g,[f,[P,P]]]] = \langle df\wedge dg\wedge dh, [P,P]\rangle$ the result follows. qed.

Our next aim is to study two actions of $\Gamma(\Lambda TM)$ on the space $\Omega(M)$ of differential forms and to express the Schouten-Nijenhuis bracket by them.

1.5. We have already used the $C^\infty(M)$-bilinear (fibrewise) pairing $\langle\ ,\ \rangle: \Omega(M) \times \Gamma(\Lambda TM) \to C^\infty(M)$, given by $\langle \omega_1\wedge\ldots\wedge\omega_p, X_1\wedge\ldots\wedge X_p\rangle = \det(\omega_i(X_j))$. For each $C^\infty(M)$-linear (tensorial) mapping $F: \Omega^p(M) \to \Omega^q(M)$ we have an adjoint $F^*: \Gamma(\Lambda^q TM) \to \Gamma(\Lambda^p TM)$ and conversely. For $\omega$ in $\Omega^p(M)$ consider the $C^\infty(M)$-linear mapping $\mu(\omega): \Omega^k(M) \to \Omega^{k+p}(M)$, $\mu(\omega)\psi = \omega\wedge\psi$ . It's adjoint willbe denoted by $\bar{i}(\omega) := \mu(\omega)^*: \Gamma(\Lambda^m TM) \to \Gamma(\Lambda^{m-p}TM)$. Likewise for $U$ in $\Gamma(\Lambda^p TM)$ consider the $C^\infty(M)$-linear mapping $\bar{\mu}(U): \Gamma(\Lambda^k TM) \to \Gamma(\Lambda^{k+p}TM)$, $\bar{\mu}(U)V = U\wedge V$. Its adjoint is denoted by $i(U) := \mu(U)^*: \Omega^m(M) \to \Omega^{m-p}(M)$. Other common notations for these two mappings are $U \lrcorner\, \omega = i(U)\omega$ , $\omega \llcorner U = \bar{i}(\omega)U$.

1.6. Lemma: Let $U$ be in $\Gamma(\Lambda^p TM)$. Then $i(U): \Omega(M) \to \Omega(M)$ is a homogeneous $C^\infty(M)$-module homomorphism of degree $-p$, the graded commutator $[i(U), i(V)] = 0$, $i(U)$ is a graded derivation of $\Omega(M)$ if and only if the degree of $U$ is 1. For $\omega$ in $\Omega^1(M)$ and $\psi$ in $\Omega(M)$ we have: $i(U)(\omega\wedge\psi) = i(\bar{i}(\omega)U)\psi + (-1)^p\omega\wedge i(U)\psi$, that is $[i(U), \mu(\omega)] = i(\bar{i}(\omega)U)$ in $\mathrm{End}(\Omega(M))$.
Finally $i(U\wedge V) = i(V)\circ i(U)$.

Proof: Put $U = X_1\wedge\ldots\wedge X_p$, apply $i(U)$ to $\omega_1\wedge\ldots\wedge\omega_q$ , evaluate at $Z_1\wedge\ldots\wedge Z_{q-p}$ and expand the determinant by the first line. Then $i(U)(\omega\wedge\psi) = i(\bar{i}(\omega)U)\psi + (-1)^p\omega\wedge i(U)\psi$ follows. The rest is easy. qed.

1.7. The Lie differential operator: For $U$ in $\Gamma(\Lambda^p TM)$ we define $\Theta(U): \Omega(M) \to \Omega(M)$ by $\Theta(U) := [i(U),d] = i(U)d - (-1)^p d\, i(U)$. Then $\Theta(U)$ is homogeneous of degree $1-p$ and is called the Lie differential operator along $U$. It is a derivation if and only if $U$ is a vector field. Note that $[\Theta(U), d] = 0$ by the graded Jacobi identity.

1.8. Lemma: 1. For $U$ in $\Gamma(\Lambda^u TM)$ and $V$ in $\Gamma(\Lambda^v TM)$ we have

$\Theta(U \wedge V) = i(V)\Theta(U) + (-1)^u\, \Theta(V)i(U)$.

2. $\Theta(X_1 \wedge \ldots \wedge X_p) = \Sigma\, (-1)^{j-1}\, i(X_p)\ldots i(X_{j+1})\Theta(X_j)i(X_{j-1})\ldots i(X_1)$.

3. $\Theta(f) = [i(f),d] = [\mu(f),d] = -\mu(df)$.

Proof: 1 follows from 1.6 and the definition. 2 follows by induction on p. 3 is clear. qed.

1.9. Theorem: For $U$ in $\Gamma(\Lambda^u TM)$ and $V$ in $\Gamma(\Lambda^v TM)$ we have:

1. $[\Theta(U),\, i(V)] = (-1)^{(u-1)(v-1)}\, i([U,V]) = i(-[V,U])$.
2. $[\Theta(U),\, \Theta(V)] = (-1)^{(u-1)(v-1)}\, \Theta([U,V]) = \Theta(-[V,U])$.

Proof: Using the graded Jacobi identity the following formula is easy to check:

(3) $[\Theta(U),\, i(V)] = (-1)^{v-1}[i(U),\, \Theta(V)] = -(-1)^{(u-1)(v-1)}[\Theta(V),\, i(U)]$.

Then 1 can be proved by induction on u+v, using (3) and all the results above, in particular 1.8.1. 2 follows also by induction on u+v. qed.

1.10. Let $U$ be in $\Gamma(\Lambda^u TM)$, $V$ in $\Gamma(\Lambda^v TM)$ and $\omega$ in $\Omega^{u+v-2}(M)$. Then we have

$\Theta(-[V,U]) = [\Theta(U),\Theta(V)] = \Theta(U)\Theta(V) - (-1)^{(u-1)(v-1)}\Theta(V)\Theta(U)$.

$\Theta(U)\Theta(V)\omega = (i(U)d - (-1)^u d\, i(U))(i(V)d - (-1)^v d\, i(V))\omega = i(U)di(V)d\omega + 0.$

$= \langle di(V)d\omega,\, U\rangle$ .

Similarly we get $\Theta(V)\Theta(U)\omega = \langle di(U)d\omega,\, V\rangle$ .

$\Theta(-[V,U])\omega = i(-[V,U])d\omega - (-1)^{u+v-1}di(-[V,U])\omega = \langle d\omega,\, -[V,U]\rangle$ .

Putting everything together, we have the following

Lemma: For $U$ in $\Gamma(\Lambda^u TM)$, $V$ in $\Gamma(\Lambda^v TM)$ and $\omega$ in $\Omega^{u+v-2}(M)$ we have:

$\langle d\omega, -[V,U]\rangle = \langle di(V)d\omega,\, U\rangle - (-1)^{(u-1)(v-1)}\langle di(U)d\omega,\, V\rangle$ .

This formula suffices to compute $[U,V]$ in coordinates, and it remains valid, if we insert any closed form $\psi$ in $\Omega^{u+v-1}(M)$ instead of $d\omega$.

This formula is the starting point of Tulczyjew, who considers the bracket $[U,V]^{Tul} = -[V,U] = (-1)^{(u-1)(v-1)}[U,V]$. The coordinate version of it boils down to the definition of Schouten.

## 2. Naturality of the Schouten-Nijenhuis bracket for skew fields.

2.1. Let $f\colon M \to N$ be a smooth mapping between manifolds. We say that $U$ in $\Gamma(\Lambda^u TM)$ and $U'$ in $\Gamma(\Lambda^u TN)$ are f-related or f-dependent, if $\Lambda^u Tf.U = U' \circ f$ holds.

$$\begin{array}{ccc} \Lambda^u TM & \xrightarrow{\Lambda^u Tf} & \Lambda^u TN \\ \uparrow U & & \uparrow U' \\ M & \xrightarrow{\quad f \quad} & N \end{array}$$ .

2.2. Proposition: 1. $U$ and $U'$ as above are f-related if and only if $i(U)\circ f^* = f^*\circ i(U'): \Omega(N) \to \Omega(M)$.

2. $U$ and $U'$ are f-related if and only if $\Theta(U)\circ f^* = f^*\circ\Theta(U'): \Omega(N) \to \Theta(M)$.

3. If $U_i$ and $U_i'$ are f-related for $i = 1,2$, then also the Schouten-Nijenhuis brackets $[U_1, U_2]$ and $[U_1', U_2']$ are f-related.

Proof: 1. Let $V$ in $\Gamma(\Lambda TM)$ and $\omega$ in $\Omega(M)$ be such that $\deg U + \deg V = \deg\omega$. Then
$\langle i(U) f^*\omega, V\rangle_x = \langle (f^*\omega)_x, U_x\wedge V_x\rangle = \langle \omega_{f(x)}\circ\Lambda T_x f, U_x\wedge V_x\rangle =$
$= \langle \omega_{f(x)}, \Lambda T_x f.U_x\wedge \Lambda T_x f.V_x\rangle = \langle \omega_{f(x)}, U'_{f(x)}\wedge \Lambda T_x f.V_x\rangle = \langle (i(U')\omega)_{f(x)}, \Lambda T_x f.V_x\rangle$
$= \langle (f^* i(U')\omega)_x, V_x\rangle = \langle f^* i(U')\omega, V\rangle_x$.
2. If $U$ and $U'$ are f-related then $\Theta(U) f^* = [i(U),d] f^* = f^* [i(U'),d]$ by 1.
For the converse let $\omega$ in $\Omega^{u-1}(N)$. Then we have $\Theta(U) f^*\omega = i(U) f^* d\omega$ and $f^*\Theta(U')\omega = f^* i(U') d\omega$. Since the $d\omega$, $\omega$ in $\Omega^{u-1}(N)$, generate $\Omega^u(N)$ over $C^\infty(N)$, we have $i(U) f^*|_0\Omega^u(N) = f^* i(U')|\,\Omega^u(N)$. This implies, using the proof of 1 for $V = 1$, that $U$ and $U'$ are f-related. .
3. This follows immediately from 2 and 1.9.2. qed.

2.3. For a vector field $X$ on $M$ let $Fl_t^X$ denote the local flow of $X$. For a multivector field let the Lie derivative of $U$ along $X$ be given by $\Theta(X)U = \frac{d}{dt}|_0(Fl_t^X)^*U$.

Lemma: $\Theta(X)U = \frac{d}{dt}|_0 (Fl_t^X)^* U = [X,U]$.

This result and the "right" signs in 1.2 convinced me to change the sign of the Schouten-Nijenhuis bracket.

Proof: It suffices to take $U = X_1\wedge\ldots\wedge X_p$, since they generate locally $\Gamma(\Lambda TM)$ over $\mathbb{R}$. But then $\Theta(X)(X_1\wedge\ldots\wedge X_p) = \frac{d}{dt}|_0 (Fl_t^X)^*(X_1\wedge\ldots\wedge X_p) =$
$= \frac{d}{dt}|_0 (Fl_t^X)^* X_1\wedge\ldots\wedge(Fl_t^X)^* X_p = \Sigma X_1\wedge\ldots\wedge[X,X_i]\wedge\ldots\wedge X_p = [X, X_1\wedge\ldots\wedge X_p]$ . qed.

2.4. An immediate consequence of the last two results is, that the Schouten-Nijenhuis bracket commutes with Lie derivatives along vector fields: $\Theta(X)[U,V] = [\Theta(X)U,V] + [U,\Theta(X)V]$. By 2.3 this is just a special case of the graded Jacobi identity.

2.5. Now we want to determine all natural concomitants of the Schouten-Nijenhuis type. So let us consider, for each n-dimensional manifold $M$, a $\mathbb{R}$-bilinear operator $B_M: \Gamma(\Lambda^p TM)\times\Gamma(\Lambda^q TM)\to\Gamma(\Lambda^r TM)$ such that for each local diffeomorphism $f:M\to N$ we have $f^*B_N(U,V) = B_M(f^*U,f^*V)$, where $(f^*U)(x) = (\Lambda^p T_x f)^{-1}U(f(x))$.
Then each $B_M$ is a local operator, and by the bilinear version of Peetre's theorem (see [1]) or the nonlinear version of it (see [11]) $B_M$ is a bilinear differential operator. So we may differentiate through $B = B_M$: given any vector field $X$ on $M$ we have $\Theta(X)B(U,V) = \frac{d}{dt}|_0 (Fl_t^X)^*B(U,V) = \frac{d}{dt}|_0 B((Fl_t^X)^*U,(Fl_t^X)^*V) =$

$= B(\Theta(X)U,V) + B(U,\Theta(X)V)$. In view of 2.3 this looks like:

$$[X, B(U,V)] = B([X,U],V) + B(U, [X,V]).$$

By naturality it suffices to determine B on $\mathbb{R}^n$.

2.7. Lemma: Let $I = \Sigma x^i(\partial/\partial x^i)$ be the identity vector field on $\mathbb{R}^n$.

1. If U is a constant p-vector field on $\mathbb{R}^n$, then $[I,U] = -p.U$.
2. For any U in $\Gamma(\Lambda^p T\mathbb{R}^n)$ we have $[I,U](0) = -p.U(0)$.
3. Let U in $\Gamma(\Lambda^p T\mathbb{R}^n)$ be a p-vector field which is homogeneous of degree k, then $[I,U] = (k-p)U$.

Proof: 2. If U = X is vector field this is easily checked. But then $[I, X_1\wedge\ldots\wedge X_p](0) = (\Sigma\ X_1\wedge\ldots\wedge[I,X_i]\wedge\ldots\wedge X_p)(0) = -p.(X_1\wedge\ldots\wedge X_p)(0)$.
3. Let first U = X be a k-homogeneous vector field. The flow of I is given by $Fl_t^I(x) = e^t.x$. So $(Fl_t^I)^*X(x) = T(Fl_t^I)^{-1}\circ X\circ Fl_t^I(x) = e^{-t}.X(e^t.x) = e^{(k-1)t}.X(x)$. Thus $[I,X] = \Theta(I)X = \frac{d}{dt}\big|_0 (Fl_t^I)^*X = \frac{d}{dt}\big|_0 e^{(k-1)t}.X = (k-1)X$. Now suppose that $U = X_1\wedge\ldots\wedge X_p$ where $X_1$ is k-homogeneous and the other $X_i$ are constant. Then $[I, X_1\wedge\ldots\wedge X_p] = \Sigma\ X_1\wedge\ldots\wedge[I,X_i]\wedge\ldots\wedge X_p = (k-p)\ X_1\wedge\ldots\wedge X_p$ by 1. qed.

2.8. Now let U in $\Gamma(\Lambda^p T\mathbb{R}^n)$ be homogeneous of degree k and let V in $\Gamma(\Lambda^q T\mathbb{R}^n)$ be homogeneous of degree m. Then we have by 2.7 and 2.6:

$$-r.B(U,V)(0) = [I, B(U,V)](0) = B([I,U],V)(0) + B(U,[I,V])(0) =$$
$$= B((k-p)U,V)(0) + B(U,(m-q)V)(0) = (k+m-p-q)B(U,V)(0).$$

From this the following result is immediate.

Corollary: If $B: \Gamma(\Lambda^p TM) \times \Gamma(\Lambda^q TM) \to \Gamma(\Lambda^r TM)$ is a natural bilinear concomitant, then B is a bilinear differential operator which is homogeneous of total order p+q-r. So B is 0 if $p+q-r < 0$, is algebraic (tensorial) if $p+q = r$, and is of total order 1 if $p+q-1 = r$ (this is our case: so on $\mathbb{R}^n$ we may write $B(U,V) = B_1(dU,V) + B_2(U,dV)$, where $B_i$ are tensorial).

2.9. Theorem: Any natural bilinear concomitant $\Gamma(\Lambda^p TM) \times \Gamma(\Lambda^q TM) \to \Gamma(\Lambda^{p+q-1} TM)$ is a constant multiple of the Schouten-Nijenhuis bracket.

Remark: A look at the proof will show that any natural bilinear concomitant $\Gamma(\Lambda^p TM) \times \Gamma(\Lambda^q TM) \to \Gamma(\Lambda^{p+q} TM)$ is a constant multiple of $(U,V) \to U\wedge V$. I have not looked at the concomitants into $\Gamma(\Lambda^r TM)$, if $r < p+q-1$. But the method applied here can be used to determine them all.

Proof: Following the method of Kolar [4] we have to determine all bilinear $GL^2(n,\mathbb{R})$ -equivariant mappings $\Lambda^p\mathbb{R}^n \times \mathbb{R}^{n*} \times \Lambda^q\mathbb{R}^n \to \Lambda^{p+q-1}\mathbb{R}^n$ and their counterparts with the role of U and V exchanged. These mappings then induce the jet-expressions of the looked for concomitants between the associated bundles of the second order frame bundle of M. Equivalently we have to find all $GL^2(n,\mathbb{R})$-equivariant linear mappings in the first line of the following diagram:

$$(1)\qquad \begin{array}{ccc} \Lambda^p\mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \Lambda^q\mathbb{R}^n & \longrightarrow & \Lambda^{p+q-1}\mathbb{R}^n \\ \downarrow\uparrow \text{ Alt x Alt x Alt} & & \downarrow\uparrow \text{ Alt} \\ \overset{p}{\otimes}\mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \overset{q}{\otimes}\mathbb{R}^n & \longrightarrow & \overset{p+q-1}{\otimes}\mathbb{R}^n . \end{array}$$

Since the vertical mappings in this diagram (where Alt stands for the appropriate alternator) are all $GL^2(n,\mathbb{R})$-equivariant it suffices to determine all $GL^2(n,\mathbb{R})$-equivariant mappings in the lower line. The action of $GL^2(n,\mathbb{R}) = GL(n,\mathbb{R}) \circledS L^2_{sym}(n,n)$ on $\overset{p}{\otimes}\mathbb{R}^n$ is given by the transformation law

$$(2)\quad \bar{U}^{\alpha_1\dots\alpha_p} = U^{\beta_1\dots\beta_p}\frac{\partial\bar{x}^{\alpha_1}}{\partial x^{\beta_1}}\cdots\frac{\partial\bar{x}^{\alpha_p}}{\partial x^{\beta_p}} ,$$

and the action on $\overset{p}{\otimes}\mathbb{R}^n \otimes \mathbb{R}^{n*}$, the space of the first order partial derivatives, is given by the transformation law

$$(3)\quad \bar{U}^{\alpha_1\dots\alpha_p}{}_{,k} = U^{\beta_1\dots\beta_p}{}_{,m}\frac{\partial\bar{x}^{\alpha_1}}{\partial x^{\beta_1}}\cdots\frac{\partial\bar{x}^{\alpha_p}}{\partial x^{\beta_p}}\frac{\partial x^m}{\partial\bar{x}^k} + U^{\beta_1\dots\beta_p}\left(\frac{\partial^2\bar{x}^{\alpha_1}}{\partial x^m\partial x^{\beta_1}}\frac{\partial x^m}{\partial\bar{x}^k}\frac{\partial\bar{x}^{\alpha_2}}{\partial x^{\beta_2}}\cdots\frac{\partial\bar{x}^{\alpha_p}}{\partial x^{\beta_p}}+\dots+\frac{\partial\bar{x}^{\alpha_1}}{\partial x^{\beta_1}}\cdots\frac{\partial^2\bar{x}^{\alpha_p}}{\partial x^m\partial x^{\beta_p}}\frac{\partial x^m}{\partial\bar{x}^k}\right) .$$

The $GL^2(n,\mathbb{R})$-equivariant mappings are in particular $GL(n,\mathbb{R})$-equivariant, stemming from the embedding $GL(n,\mathbb{R}) \hookrightarrow GL^2(n,\mathbb{R})$, see (3) with all second partial derivatives 0. According to the theory of invariant tensors, as explained in Dieudonné-Carrell, the $GL(n,\mathbb{R})$-equivariant mappings are given by all permutations of the indices, all contractions and tensorizing with the identity. Permutations of indices do not play a role since we take alternator afterwards, the identity cannot appear since the result is purely contravariant. So we may just contract the derivation entry of U into the vector part of U or into V, and the same with U and V interchanged. So we have the following 4-parameter family, where $U = U^\alpha\partial_\alpha$, $V = V^\beta\partial_\beta$, $dU = U^\alpha{}_{,j}\,\partial_\alpha\otimes d^j$, $B(U,V) = B^\gamma\partial_\gamma$ .
We do not indicate alternation in the upper indices, and we write α, β for any kind of multivector index, so γ = (α,β) etc.

$$(4)\quad B^\gamma = a.\, U^{m\alpha}{}_{,m}V^\beta + b\, U^\alpha{}_{,m}V^{m\beta} + c\, U^\alpha V^{n\beta}{}_{,n} + d\, U^{n\alpha}V^\beta{}_{,n} .$$

But the mapping B is also equivariant with respect to the abelian normal subgroup $\{Id\}\times L^2_{sym}(n,n) \hookrightarrow GL^2(n,\mathbb{R})$. The action of an element $(Id,S) \doteq (Id, S^i_{jk})$ on $\overset{p}{\otimes}\mathbb{R}^n$ is the identity, on $\overset{p}{\otimes}\mathbb{R}^n\otimes\mathbb{R}^{n*}$ it is given by (using (3)):

$$(5)\quad \bar{U}^\alpha{}_{,k} = U^\alpha{}_{,k} + U^{t\alpha_2\dots\alpha_p}S^{\alpha_1}_{tk} + \dots + U^{\alpha_1\dots\alpha_{(p-1)}t}S^{\alpha_p}_{tk} .$$

So the expression (4) has to be invariant under the action of $\{Id\}\times L^2_{sym}(n,n)$ on the right hand side. This is equivalent to the following equation:

$$(6)\ 0 = a\,( U^{t\alpha} S^{m}_{tm} + U^{mt\alpha} S^{\alpha 2}_{tm} + \dots + U^{m\alpha t} S^{\alpha p}_{tm} )\, V^{\beta} + \\ + b\,( U^{t\alpha} S^{\alpha 1}_{tm} + \dots + U^{\alpha t} S^{\alpha p}_{tm} )\, V^{m\beta} + \\ + c\, U^{\alpha} ( V^{t\beta} S^{n}_{tn} + V^{nt\beta} S^{\beta 2}_{tn} + \dots + V^{n\beta t} S^{\beta q}_{tn} ) \\ + d\, U^{n\alpha} ( V^{t\beta} S^{\beta 1}_{tn} + \dots + V^{\beta t} S^{\beta q}_{tn} ).$$

This can be simplified after taking alternation:

$$(7)\ 0 = a\, U^{m\alpha} S^{t}_{mt} V^{\beta} + b\, p\, U^{\alpha m} S^{\alpha p}_{mn} V^{n\beta} + c\, U^{\alpha} S^{t}_{nt} V^{n\beta} + d\, q\, U^{m\alpha} S^{\beta 1}_{mn} V^{n\beta} .$$

Now we can compare coefficients and get the following relations:

$$a = 0,\ p.b + (-1)^{p-1} q.d = 0,\ c = 0 .$$

So there is only one parameter surviving and we get

$$B^{\gamma} = c.( q\, U^{\alpha}{}_{,m} V^{m\beta} + (-1)^{p}\, p\, U^{n\alpha} V^{\beta}{}_{,n} ) = c.( q\, U^{\alpha}{}_{,m} V^{m\beta} - p\, U^{\alpha n} V^{\beta}{}_{,n} ) \\ = c.[U,V].$$

The comparison of coefficients is valid if these tensors are really independent. If $p+q \geq \dim M$ some expressions are 0. Since they may be viewed as linear mappings in S, they are linearly independent as long as they are nonzero. And if a summand in (7) becomes 0, the corresponding one in (4) is zero also. qed.

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