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ABSTRACT

One argues here that CCR leads rather to "no joint localization principle" than to the so called "uncertainty" principle.

In prevailing majority of existing by now textbooks on Quantum Mechanics the canonical commutation relation is interpreted as leading to well known inequality for statistical dispersion, which is at the same time identified with Heisenberg uncertainty principle [6,8 9].

On the other hand, one may doubt whether it makes sense at all [3].

Other authors, for example Dirac [5] or Bohm [2], introduce the uncertainty principle via unavoidable wave packet concept, so this way of looking might be related to support properties of functions and their Fourier transforms [4].

Anyhow, in almost all formulations of "uncertainty" principle hidden variables are somehow hidden, while (as we seem to know it now well), the hidden variables concept contradicts the rules of quantum mechanics.

In the following we shall argue that CCR should be rather related to "no joint, bounded localization" - principle, then to "uncertainty" principle.

We shall call-the dispersion property - the theorem

$$\mathcal{D}_\psi C \; \mathcal{D}_\psi D \geq \frac{1}{2} |\langle\psi[C,D]|\psi\rangle|$$

with C and D observables and $|\psi\rangle$ belonging to the appropriate domain Ω , dense in Hilbert space $\mathcal{H}$. The observables are defined as selfadjoint operators in a rigged Hilbert space, with however complete set

of r e g u l a r eigenvectors.
The rigged Hilbert space setting as advocated in [1] , seems to be most convenient for our purposes.
In this context note that selfadjoint operators like X or P, whose spectrum is not discrete, are not observables; however their spectral families provide us with sets of observables of a kind, and correspond thus to various localizations in for example X or P spectra.

We start the discussion of the subject with an example.
Let us consider the minimal wave packet defined by

$$\mathcal{D}_\psi X \mathcal{D}_\psi P = \frac{\hbar}{2} \quad ,$$

and choose the normalized coherent state $|\psi\rangle$ so as to give $\langle\psi|X|\psi\rangle = \langle\psi|P|\psi\rangle = 0$, where $\mathcal{D}_\psi X$ is defined the same way as dispersion $\mathcal{D}_\psi C$ for an observable C.
Let $\{|X\rangle\}$ denotes the complete set of generalized eigenvectors of X in Ω'. Correspondingly let $\{|p\rangle\}$ denotes the same with respect to P. These two basises are chosen so as to yield

$$\langle x|y\rangle \equiv \delta(x-y) \qquad \text{and} \qquad \langle p|p'\rangle = 2\pi\hbar\delta(p-p') .$$

Hence in the $\{|x\rangle\}$ basis one has

$$\langle x|\psi\rangle = \frac{1}{(2\pi)^{\frac{1}{4}}\Delta^{\frac{1}{2}}} \exp\left\{-\frac{x^2}{4\Delta^2}\right\} \quad ,$$

with obvious notation: $\mathcal{D}_\psi X = \Delta$ and $\mathcal{D}_\psi P = \frac{\hbar}{2\Delta} \equiv \tilde{\Delta}$.
Define now two projection operators:

$$E(n\Delta) = \int_{-n\alpha}^{+n\alpha} |x\rangle\langle x| \, dx \quad , \qquad 2\alpha = \Delta, \qquad \text{and}$$

$$\tilde{E}(n\tilde{\Delta}) = \int_{-n\beta}^{+n\beta} |p\rangle\langle p| \frac{dp}{2\pi h} \quad , \qquad 2\beta = \tilde{\Delta}, \quad n \in R^+ .$$

One then readily sees that the variance $\mathcal{D}_\psi X$ could be practically identified with an almost-localization interval if one has, an approximate equality

$$\langle\psi|E(n\Delta)|\psi\rangle \simeq 1, \qquad \text{up to, say, 1\% for } n=1 .$$

This is however far to be the case, as one has for n=1

$$\langle\psi|E(\Delta)|\psi\rangle = 0.69 = \langle\psi|\tilde{E}(\tilde{\Delta})|\psi\rangle$$

and only for n=6 we get the desired, up to 1% equality

$$\langle\psi|E(6\Delta)|\psi\rangle \cong 1.$$

This particular example indicates that the idea of an eventual identification of variances $\mathcal{D}_\psi X$ and $\mathcal{D}_\psi P$ for an arbitrary wave packet $|\phi\rangle = \int_R \phi(x)|x\rangle dx$ with the almost localizations of the quantum phenomenon in respective spectra - fails.

What is then the meaning of saying that "$\Delta \equiv \mathcal{D}_\psi X$ measures uncertainty of the position measurement"?,... or what is the meaning of the statement:

(H) "$\mathcal{D}_\phi X \mathcal{D}_\phi P \geq \frac{\hbar}{2}$ expresses the limits of accuracy within which joint measurements of position and momentum are possible"?

Supposedly there are at least two assumptions in such a way of thinking.

I. At first, one assumes that a quantum phenomenon (electron, neutron, meson, etc.) <u>has</u> its own position and momentum which, however - due to (H) - cannot be stated, measured without uncertainty.

II. Secondly, one assumes in (H) that joint (in P's and X's) localization procedure is, "to some extent", perfomable.

We shall not enter here into an epistemological discussion of these views.

Instead - we shall try to argue that these assumptions seem to contradict quantum mechanics rules.

The assumption I. is the basis of a kind of hidden variables idea, which seems by now to be refuted by Gedanken experiments. According to this a term "uncertainty" is somewhat a misnomer.
Actual quantum phenomenon is completely characterized by its state, which is certain and probability distributions are completely certain too, probabilities for answers to all admissible questions. It may happen (it happens) that for particular states two properties are never actual; for example localization in an interval of P's and localization in another interval of X's. (The notion of the "ac-

tual property" we use in a sense defined in [7]).
Hence simply there is nothing to be "uncertain" in such cases.

Anyhow, since the arguments against hidden variables idea are well known we feel free to discuss the assumption II only, in what follows.

We take the (practical?) standpoint, that any p o s s i b l e localization procedure corresponds to a practically accurate preparation of a finite i n t e r v a l of R in a way depicted below

$$\underset{(-\infty,\ a]}{\overline{\qquad\qquad}} \qquad \underset{[b,\ +\infty)}{\overline{\qquad\qquad}} \quad ; \quad X'\text{s spectrum}$$

$a<b$

i.e. one constructs a slit with counterparts of two projection operators $E_L \equiv E\{(-\infty,a]\}$ and $E_R \equiv E\{[b,+\infty)\}$ from the spectral family of X. The same for P.
If the quantum phenomenon is characterized by the state $|\phi\rangle \neq 0$ such that $E_L|\phi\rangle = E_R|\phi\rangle = 0$ then we say that it has,as, its actual property, the localization [a,b].
Put it another words: whenever a detector is placed at the slit an "Yes" outcome is certain.

Consider now similar situation corresponding to localization in momentum P spectrum:

$$\underset{(-\infty,\tilde{a}]}{\overline{\qquad\qquad}} \qquad \underset{[\tilde{b},+\infty\rangle}{\overline{\qquad\qquad}} \quad /P'\text{s spectrum}$$

$\tilde{a}<\tilde{b}$

i.e. consider now the "P-slit" prepared with help of

$$\tilde{E}_L = \tilde{E}\{(-\infty,\tilde{a}]\} \qquad \text{and} \qquad \tilde{E}_R = \tilde{E}\{[\tilde{b},+\infty)\} \qquad \text{procedures.}$$

Our question resulting from (H) may be now stated in the form:

"Do the states $0 \neq |\phi\rangle$ such that

$$E_L|\phi\rangle = E_R|\phi\rangle = \tilde{E}_L|\phi\rangle = \tilde{E}_R|\phi\rangle = 0, \text{ exist ? "}$$

The answer is - no.
It follows from [4] where it was shown in particular that $\mathrm{com}(E,\tilde{E})=0$ for any projections E and $\tilde{E}$ on half-lines of the corresponding spectra.

We then see that E and E are totally noncommutative spectral projections.
This fact should be then interpreted rather as

"no joint, bounded localization principle"

and we do not see any way it might be related with an uncertainty of some properties of quantum phenomenon.

The $[X,P]=i\hbar$ relation has of course more meaning then that; however, all its physical interpretation seems to be confined to properties of X's and P's spectral families - as only lattice of projections (lattice of properties [7]) seems to have well defined operational meaning.
Our conclusion then is, that the (H) statement, in the conventional framework can hardly be given a meaning.

In [3] one discusses a possible precisation of (H) statement in terms of the so called unsharp position and momentum observables.
If these are to be however observables yielding quantum properties, then we face again objections which make us to reject the assumption I.
Our point of view relies on the conviction (postulate) that questions for sharp position and sharp momentum belong only to classical property lattice, independently whether these are accompanied by some probability density functions expressing "unsharpness of measuring devices" - or not.
We think also, that an accurate interpretation of property lattice for a quantum phenomenon should not allow to call X or P "observables" because questions corresponding to single points of their spectra do not exist.
Neither it seems to make sense to consider

$\mathcal{D}_{\phi}X\mathcal{D}_{\phi}P \geqq \frac{\hbar}{2}$ relation as the statistical dispersion property

because variances $\mathcal{D}_{\phi}X$ and $\mathcal{D}_{\phi}P$ are not dispersions of realizable outcomes.
At the end let us come back again to the meaning of CCR in the context of joint localizations.
Namely we want to indicate, following [4] and its reference [23], that there exist joint localizations in position and momenta for any state of quantum phenomenon as far as localizations within periodic nontrivial Borel sets are concerned.
This is what might be considered to correspond to the situation of

an "ideal" crystal.

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