SOME GENERALIZATION OF GODEMENT'S THEOREM ON DIVISION

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ABSTRACT. Some generalization of Godement's theorem on division is found. This generalization characterizes all equivalence relation R (on a C^{∞}-manifold) such that every abstract class of R has a countable number of arcwise connected components and the family of all such components is a foliation. Using it, another proof of that classical Godement's theorem is obtained.

The classical Godement's theorem on division [3] - which characterizes regular equivalence relations R on a C^{∞}-manifold V - is well known:

THEOREM 1. (Godement [3]). Let $\dim V=n$. The following conditions are equivalent:

(1) In the set $V/_R$ there exists a differential structure of an n-k-dim. C^{∞}-manifold (with the quotient topology), such that the natural projection $V \to V/_R$ is a submersion.

(2) (a) $R\subset V\times V$ is a proper n+k-dim. C^{∞}-submanifold of $V\times V$,

(b) $pr_1 : R \to V$, $(x,y) \mapsto x$, is a submersion. ∎

The family $\mathcal{L}$ of all abstract classes of an equivalence relation R fulfilling (1) has the following properties:

(1^{o}) every abstract class of R has a countable number of arcwise connected components,

(2^{o}) the family $\mathcal{F}$ of all arcwise connected components of all abstract classes of R is a k-dim. foliation.

Of course, here: each arcwise connected component is equal to a connected component.

Now, we give some generalization of Godement's theorem which characterizes all equivalence relations fulfilling (1^{o}) and (2^{o}) (in particular, all foliations).

THEOREM 2. Let R be any equivalence relation on a Hausdorff C^{∞}-manifold V with a countable basis. The following conditions are equivalent:

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(1) the family $\mathcal{L}$ of all abstract classes of R has the above properties (1^o) and (2^o),

(2) there exists a subset $\Omega \subset R$ such that

(i) $\Delta \subset \Omega$ where $\Delta = \{(x,x);\ x \in V\}$,

(ii) Ω is a proper n+k-dim. C^∞-submanifold of $V \times V$,

(iii) $pr_1 | \Omega : \Omega \to V$ is a submersion,

(iv) if we denote, for $(x,y) \in R$,
$R_x := R \cap (\{x\} \times V)$, $\Omega_x := (pr_1 | \Omega)^{-1}(x)$, $D_{(x,y)} : R_y \to R_x$, $(y,t) \mapsto (x,t)$, then we have that the set $D_{(x,y)}[\Omega_y] \cap \Omega_x$ is open in the man. Ω_x,

(v) the manifolds $\tilde{R}_x$ (see lemma below) have a countable number of connected components.

<u>LEMMA</u>. If $\Omega \subset R$ has properties (i)÷(iv), then, for each point $x \in V$, there exists exactly one C^∞-manifold $\tilde{R}_x$ with the set of points R_x, such that, for each $(x,y) \in R_x$,

(a) $D_{(x,y)}[\tilde{\Omega}_y] \subset \tilde{R}_x$ (i.e. is open in $\tilde{R}_x$),

(b) $D_{(x,y)} | \Omega_y : \Omega_y \to D_{(x,y)}[\Omega_y] \subset \tilde{R}_x$ is a diffeomorphism.

The manifolds $\tilde{R}_x$ have the properties:

(i) $D_{(x,y)} : \tilde{R}_y \to \tilde{R}_x$ is a diffeomorphism,

(ii) $\tilde{R}_x \hookrightarrow V \times V$ is an immersion,

(iii) $\tilde{R}_x$ are Hausdorff,

(iv) if, in addition, the family $\mathcal{F}$ of all arcwise connected components of all abstract classes of R is a k-dim. foliation, then the mapping $\gamma_x : L_x \to \tilde{R}_x$, $y \mapsto (x,y)$, is a diffeomorphism for each $x \in V$ (L_x - the abstract class of R through x equipped with the uniquely determined differential structure of an immerse submanifold of V such that each element of $\mathcal{F}$ contained in L_x is an open subman. of L_x).

The very simple <u>proof</u> of this lemma is omitted.

<u>Proof of theorem 2</u>. <u>(1)⇒(2)</u>. Let us take any nice covering $\{(U_i, \varphi_i, \mathbb{R}^n);\ i \in N\}$ of $\mathcal{F}$ [2,p.188] and denote by Q^i_x the plaque of the chart (U_i, φ_i) which contains x, $x \in U_i$. Of course, the covering $\{U_i, i \in N\}$ of V has the property:

(*) if $x, y \in U_i \cap U_j$ and $y \in Q^i_x$, then $y \in Q^j_x$, $i, j \in N$.

We put
$$\Omega_i := \{(x,y) \in V \times V;\ x \in U_i,\ y \in Q^i_x\} \quad \text{and} \quad \Omega := \bigcup_{i \in N} \Omega_i .$$

We prove that Ω has properties (i)÷(v). (i) is evident. To prove (ii), it suffices to show that

1^o) Ω_i is open in Ω (with respect to the topology induced from $V \times V$),

2^o) Ω_i is a proper C^∞-submanifold of $V \times V$.

1^o) results from the equality $\Omega_i = \Omega \cap (U_i \times U_i)$ which is a consequence of (*). To show 2^o), we first define, for each chart (U_i, φ_i), the mappings φ_i^1 and φ_i^2 in such a way that

$$\varphi_i = (\varphi_i^1, \varphi_i^2) = (x \longmapsto (\varphi_i^1(x), \varphi_i^2(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k}).$$

Next, we put

$$v_i : \Omega_i \longrightarrow \mathbb{R}^{n+k},\ (x,y) \longmapsto (\varphi_i^1(x),\ \varphi_i^2(x),\ \varphi_i^1(y)).$$

The inverse mapping of v_i is

$$w_i : \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}^k \longrightarrow \Omega_i,\ (a,b,c) \longmapsto (\varphi_i^{-1}(a,b),\ \varphi_i^{-1}(c,b)).$$

Now, it is easy to see that 2^0) holds (v_i is a global chart on Ω_i). To show condition (iii), it is enough to consider the following commuting diagram

$$\begin{array}{ccc} \Omega_i & \xrightarrow[\approx]{v_i} & \mathbb{R}^n \times \mathbb{R}^k \\ {\scriptstyle pr_1|\Omega_i}\downarrow & & \downarrow{\scriptstyle pr_1} \\ U_i & \xrightarrow[\approx]{\varphi_i} & \mathbb{R}^n \end{array}$$

for each $i \in N$. To notice condition (iv), we write $\Omega_x = \{x\} \times \bigcup_{i \in N_x} Q_x^i$ where $N_x := \{i \in N;\ x \in U_i\}$. Therefore

$$D_{(x,y)}[\Omega_y] \cap \Omega_x = \bigcup_{j \in N_y,\, i \in N_x} (\{x\} \times (Q_y^j \cap Q_x^i)) \subset \Omega_x.$$

Condition (v) follows from property (iv) of $\tilde{R}_x$ from our lemma.

<u>(2)⇒(1)</u>. We assume that $\Omega \subset R$ fulfils (i)÷(v). Let us take the embedding $\hat{u} : V \longrightarrow \Omega$, $x \longmapsto (x,x)$. Of course, $\hat{u}^* T^\alpha \Omega$ (where $T^\alpha \Omega = \operatorname{Ker} \alpha_*$, $\alpha = pr_1|\Omega$) is a vector bundle of rank k over V. We define a strong homomorphism $\varkappa$ of vector bundles as a superposition

$$\begin{array}{ccccccccc} \hat{u}^* T^\alpha \Omega & \longrightarrow & T^\alpha \Omega & \longrightarrow & T\Omega & \longrightarrow & T(V \times V) & \xrightarrow{(pr_2)_*} & TV \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow{\hat{u}} & \Omega & = & \Omega & \hookrightarrow & V \times V & \xrightarrow{pr_2} & V. \end{array}$$

$\varkappa$ is a monomorphism because, for $x \in V$, $\varkappa_{|x}(v) = (j_x)_*(v)$, $v \in T_{(x,x)}\Omega_x$, where $j_x : \{x\} \times V \xrightarrow{\approx} V$, $(x,y) \longmapsto y$. Thus $E := \operatorname{Im} \varkappa \subset TV$ is a vector subbundle of order k of TV, and $E_{|x} = (j_x)_{*(x,x)}[T_{(x,x)}\tilde{R}_x]$, $x \in V$.

Via bijections $\gamma_x : L_x \rightarrow \tilde{R}_x$, $y \longmapsto (x,y)$, $x \in V$, every abstract class of R is equipped with a differential structure of a manifold. The correctness follows from property (i) of the manifolds $\tilde{R}_x$ (see lemma). The manifolds obtained are integral for the distribution E. Indeed, for $x \in V$, the inclusion $L_x \hookrightarrow V$ is an immersion (because it is the superposition $L_x \xrightarrow{\gamma_x} \tilde{R}_x \hookrightarrow \{x\} \times V \xrightarrow{j_x} V$), and $T_x L_x = (j_x)_{*(x,x)}[T_{(x,x)}\tilde{R}_x] = E_{|x}$. Let $\mathcal{F}$ be the family of all connected components of all manifolds L_x obtained above. By the Frobenius' theorem [1,p.86], $\mathcal{F}$ is a k-dim. foliation. To conclude this theorem, we need to demonstrate that the family $\mathcal{F}$ is equal to the family of arcwise connected components of all abstract classes of R. For the purpose, it is sufficient to show that every manifold L_x is a k-leaf of V with respect to all locally arcwise connected topological spaces, i.e. if X is such a space and $f : X \longrightarrow V$ a continuous mapping such that $f[X] \subset L_x$, then the induced mapping $f : X \longrightarrow L_x$ is continuous, too. Let X and f be as above; take $t\in X$ and (U,φ) - a chart around $y:=f(t)$ distinguished by $\mathcal{F}$, $\varphi:U\longrightarrow \mathbb{R}^k\times\mathbb{R}^{n-k}$. Let Q be an arcwise connected component of $U\cap L_x$ through y(with respect to the topology induced from V). Q contains countably many plaques of the chart (U,φ) since L_x has - by (v) - countably many connected components, and each of them - as a connected immerse submanifold of V - has a countable basis. Thus $pr_2[\varphi[Q]]$ is an arcwise connected and countable set in $\mathbb{R}^{n-k}$, so it is one-point. This states that Q is equal to one plaque of the chart (U,φ). The set $f^{-1}[U\cap L_x]$ is open in X. Let B be the arcwise connected component of the set $f^{-1}[U\cap L_x]$, containing x. Of course, B is open in X, $f[B]\subset Q$ and $f|B:B\longrightarrow V_{|Q}=L_{x|Q}$ is continuous. The free choice of $t\in X$ implies the continuity of $f:X\longrightarrow L_x$. ∎

REMARK. The connectivity of the manifolds $\tilde{R}_x$ is equivalent (in the above theorem) to the fact that $\mathcal{L}$ is a foliation(i.e. every abstract class of R is an arcwise connected set). ∎

REMARK. If Ψ is the subgroupoid (of the groupoid determined by R) generated by the set Ω fulfilling conditions (i)÷(iv) from theorem 2, then the set $\Psi\cap(\{x\}\times V)$, $x\in V$, ia an open-closed subset of $\tilde{R}_x$. ∎

THEOREM 3. The following conditions are equivalent:

(1) the family of all abstract classes of R is a k-dim. foliation,

(2) there exists a subset $\Omega\subset R$ such that

(i)÷(iv) as in theorem 2,

(v′) Ω generates R (as a groupoid),

(vi′) the manifolds Ω_x are connected.

Proof. (1)⇒(2). The set Ω constructed in the proof of theorem 2 fulfils (vi′) in an evident manner. The connectedness of manifolds $\tilde{R}_x$ implies that (v′) follows from the last remark.

(2)⇒(1). It suffices to show that the manifolds $\tilde{R}_x$ are connected. Let us take any $x_0\in V$ and $y\in L_{x_0}$. Since Ω generates R, there exist points $x_1,\dots,x_{n-1}\in L_{x_0}$ such that (in the groupoid R) $(x_0,y)=$ $=(x_{n-1},y)\cdot\ldots\cdot(x_1,x_2)\cdot(x_0,x_1)$ where $(x_i,x_{i+1})\in\Omega_{x_i}$, $i=0,\dots,n-1$, $x_n=y$. (vi′) implies the existence of curves $c_i:\langle 0,1\rangle\longrightarrow\Omega_{x_i}$ such that $c_i(0)=(x_i,x_i)$, $c_i(1)=(x_i,x_{i+1})$. We define a curve $c:\langle 0,n\rangle\longrightarrow\tilde{R}_{x_0}$ by the formula $c(t)=D_{(x_0,x_i)}(c_i(t-i))$ for $i\leqslant t\leqslant i+1$ to obtain a continuous curve joining (x_0,x_0) and (x_0,y). ∎

Theorem 2 enables us to carry out another proof of the classical Godement's theorem.

Proof of theorem 1. (1)⇒(2) as in Godement's proof.

(2)⇒(1). Let us suppose (a) and (b). We notice that $\Omega:=R$ fulfils properties (i)÷(v) from assertion (2) in theorem 2. Thus, theorem 2 states that the family $\mathcal{F}$ of all connected components of all abstract classes of R is a k-dim. foliation.

Now, we prove that, for each point $x \in V$, there exist a real number $a>0$ and a chart (U,φ) around x distinguished by $\mathcal{F}$, such that

(i) $\varphi: U \xrightarrow{\cong} \mathbb{R}^k \times K(a)$ where $K(a) := \prod^{n-k}(-a,a)$,

(ii) $\varphi(x) = (0,0)$,

(iii) if L is an abstract class of R and $L \cap U \neq \emptyset$, then $L \cap U$ is exactly one plaque of the chart (U,φ).

Let us assume to the contrary that there exists a point $x_o \in V$ such that, for each real number $a>0$ and each chart (U,φ) around x_o distinguished by $\mathcal{F}$, fulfilling (i) and (ii), we have: there is an abstract class L of R such that the set $L \cap U$ contains at least two different plaques. Take any chart (U,φ) around x_o distinguished by $\mathcal{F}$ such that $\varphi: U \xrightarrow{\cong} \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\varphi(x_o) = (0,0)$. Let us set

$$U_m := \varphi^{-1}[\mathbb{R}^k \times \prod^{n-k}(-\tfrac{1}{m},\tfrac{1}{m})], \quad m \in \mathbb{N}.$$

Of course, $(U_m, \varphi|U_m)$ is a chart distinguished by $\mathcal{F}$, too. Then we find an abstract class L_m such that $L_m \cap U_m$ contains two plaques Q^1_m and Q^2_m, say $Q^1_m := \varphi^{-1}[\mathbb{R}^k \times \{c^1_m\}]$, $Q^2_m := \varphi^{-1}[\mathbb{R}^k \times \{c^2_m\}]$, for some $c^1_m \neq c^2_m$. Let us put $x^s_m := \varphi^{-1}(0, c^s_m)$, $s=1,2$. Of course, $x^s_m \in L_m$, which means that $(x^1_m, x^2_m) \in R$. Besides $x^s_m \xrightarrow[m\to\infty]{} x_o$, $s=1,2$. Take

$$\Omega' := \{(x,y) \in V \times V;\ x \in U \text{ and } y \in Q_x\}$$

where Q_x denotes the plaque of (U,φ) through x. We prove that Ω' is open in R. First, we note (as about Ω_i in the proof of theorem 2) that Ω' is a proper n+k-dim. C^∞-submanifold of $V \times V$. Thus Ω' is an n+k-dim. proper submanifold of the n+k-dim. manifold R, so it is open in R. Further, since $(x^1_m, x^2_m) \notin \Omega'$ and $(x^1_m, x^2_m) \xrightarrow[m\to\infty]{} (x_o, x_o)$, therefore $(x_o, x_o) \in \Omega'$, which leads to a contradiction because $x_o \in Q_{x_o}$ implies $(x_o, x_o) \in \Omega$.

From the above it follows that there exists a C^∞-atlas on V consisting of some chart (U,φ) distinguished by $\mathcal{F}$ such that (i)÷(iii) hold for $a = a_\varphi$. Let $\mathcal{A}$ be such an atlas. With the help of $\mathcal{A}$, we shall construct a C^∞-atlas on the topological space V/R, such that the projection $\pi: V \to V/R$ is a submersion. First, from the equality $\pi^{-1}[\pi[U]] = pr_2[pr_1^{-1}[U]]$, $U \subset V$, we get the openess of the projection π. Next, taking a chart $(U,\varphi) \in \mathcal{A}$, we define $\tilde{\varphi}: \tilde{U} \to K(a_\varphi)$, where $\tilde{U} = \pi[U]$, in such a way that the diagram

$$(*)\qquad \begin{array}{ccc} U & \xrightarrow{\ \pi\ } & \tilde{U} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\tilde{\varphi}} \\ \mathbb{R}^k \times K(a_\varphi) & \xrightarrow{\ p_2\ } & K(a_\varphi), \end{array} \qquad p_2(x,y) = y,$$

comutes. Of course, we must put $\tilde{\varphi}(L) := p_2(\varphi(x))$ for $x \in U \cap L$, $L \in \tilde{U}$. The correctness follows from the fact that $U \cap L$ contains exactly one plaque. The continuity of $\tilde{\varphi}$ follows from the equality $\tilde{\varphi}^{-1}[A] =$

$=\pi[\varphi^{-1}[\mathbb{R}^k\times A]]$, $A\subset K(a_\varphi)$, whereas the openness - from $\tilde{\varphi}[B]=p_2[\varphi[\pi^{-1}[B]\cap U]]$, $B\subset\tilde{U}$. The bijectivity of $\tilde{\varphi}$ is evident. In the end, we take two charts (U,φ) and $(W,\psi)\in\mathcal{A}$ such that $\tilde{U}\cap\tilde{W}\neq\emptyset$. We prove that $\tilde{\psi}\circ\tilde{\varphi}^{-1}$ is of C^{∞}-class. For the purpose, we put $\Theta:=\pi^{-1}[\tilde{U}\cap\tilde{W}]\subset V$. We notice that Θ is saturated by abstract classes of R, and $\pi[\Theta]=\pi[\Theta\cap U]=\pi[\Theta\cap W]=\tilde{U}\cap\tilde{W}$. Now, we prove - auxiliarily - that $\tilde{\varphi}|\tilde{U}\cap\tilde{W}\circ\pi|\Theta: V_{|\Theta}\to\tilde{\varphi}[\tilde{U}\cap\tilde{W}]\subset\mathbb{R}^{n-k}$ is a submersion. In order to do this, we consider the diagram

$$(**)\qquad \begin{tikzcd} ((\Theta\cap U)\times V)\cap R \arrow[r, "\text{sub.}", "pr_1"'] \arrow[d, "pr_2"', "\text{sub.}"] & \Theta\cap U \arrow[d, "\text{sub.}"', "\tilde{\varphi}\circ\pi|\Theta\cap U"] \\ \Theta \arrow[r, "\tilde{\varphi}\circ\pi|\Theta"] & \tilde{\varphi}[\tilde{U}\cap\tilde{W}] \end{tikzcd}\qquad (\text{sub.} = \text{submersion})$$

From $(*)$ we get the submersivity of $\tilde{\varphi}\circ\pi|\Theta\cap U=\tilde{\varphi}|\tilde{U}\cap\tilde{W}\circ\pi|\Theta\cap U$, whereas from diagram $(**)$ - the submersivity of $\tilde{\varphi}\circ\pi|\Theta$. Changing φ to ψ, we get the smoothness of $\tilde{\psi}\circ\pi|\Theta$. To prove that of $\tilde{\psi}\circ\tilde{\varphi}^{-1}$, it is sufficient to analyse the diagram below:

$$\begin{tikzcd} & & \tilde{\psi}[\tilde{U}\cap\tilde{W}] \\ \Theta \arrow[r, "\pi|\Theta"] \arrow[urr, bend left, "\tilde{\psi}\circ\pi|\Theta"] \arrow[drr, bend right, "\tilde{\varphi}\circ\pi|\Theta"'] & \tilde{U}\cap\tilde{W} \arrow[ur, "\tilde{\psi}"] \arrow[dr, "\tilde{\varphi}"'] & \\ & & \tilde{\varphi}[\tilde{U}\cap\tilde{W}] \arrow[uu, "\tilde{\psi}\circ\tilde{\varphi}^{-1}"'] \end{tikzcd} \quad . \ \blacksquare$$

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