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KAC-MOODY ALGEBRAS : AN INTRODUCTION FOR PHYSICISTS

David I. Olive

1 Introduction

The aim of these notes is to provide an introduction to the theory of Kac-Moody algebras accessible to theoretical physicists. It is based on the first three of five lectures given at the Srni Winter School (the last two lectures dealt with vertex operators along the lines of the November 1983 Berkeley lectures written up by Peter Goddard and myself; (GODDARD and OLIVE 1983)). These notes will be complemented by those of Peter Goddard also given at Srní (GODDARD). I am grateful to our hosts from the Charles University in Prague for their hospitality in providing such beautiful and comfortable surroundings.

2 Relevance of Kac-Moody algebras and their development

Roughly speaking the theory provides a precise mathematical framework for studying quantum field theories in two space time dimensions where the "Schwinger term" plays a crucial role. Because of this there were three apparently independent developments in theoretical particle physics during the 1960's which now have important influences on the mathematical theory. These are

(a) Skyrme's fermion-boson equivalence constructing a fermionic quantum field operator for the Sine-Gordon soliton; (SKYRME).
(b) Gellmann's "quark model" construction of current algebra; (GELLMANN; GELLMANN and NEEMAN; ADLER and DASHEN) : the use of current algebra to determine dynamics by expressing the (traceless) energy momentum tensor in terms of the currents, as proposed by Sugawara and others (SOMMERFIELD, SUGAWARA).
(c) The vertex operator construction of physically realistic particle scattering amplitudes in string theory (NAMBU; FUBINI, GORDON and VENEZIANO; FUBINI and VENEZIANO).

In 1968 Victor Kac and Bob Moody enumerated a class of infinite dimensional Lie algebras which were nevertheless tractable because the dimension diverged in the mildest possible way (KAC 1968; KAC 1983; MOODY). These algebras possessed an integer grading respected by the commutation relations such that the dimension of the subspace with a given grade did not increase too fast as the label diverged (and turned out to be bounded).

As will be explained here it proved possible to construct the root system of the algebra rather analogously to that of the ordinary finite dimensional simple Lie algebras. Further, certain representations could be found similarly, together with formulae for their characters, generalising Weyl's famous expression. Just as in the finite dimensional case the denominator can be written in two alternative ways now involving either infinite products or infinite sums. Thus an algebraic theory subsumed and generalised all sorts of identities concerning theta functions as studied in nineteenth century complex analysis, and so in mathematics too, unexpected and gratifying unifications occurred.

After about 1978 the developments in theoretical physics (a) (b) and (c) mentioned above were incorporated into the representation theory of Kac-Moody algebras showing that fascinating possibilities could occur with no parallels in ordinary Lie algebra theory. These are explained in the Berkeley notes already mentioned and in Peter Goddard's lectures (GODDARD and OLIVE 1983, GODDARD).

Very recently there has been a feedback into theoretical physics enriching three apparently diverse areas :-

(i) The theory of integrable field equations and solitons (particularly in the work of the Kyoto school (JIMBO and MIWA; see also OLIVE and TUROK 1983b,1984).
(ii) The theory of critical phenomena in two-dimensional lattice theories (FRIEDAN, QIU and SHENKER; GODDARD and OLIVE 1984; GODDARD, KENT and OLIVE).
(iii) The construction of consistent and realistic quantum string theories of all the particle interactions; (GREEN and SCHWARZ; GROSS et al).

Many other exciting developments can undoubtedly be expected in physics.

As we have explained the theory blends together in a remarkable way, nineteenth century analytic function theory, two dimensional quantum field theory and the algebraic theory of roots and weights of Cartan, Weyl, Dynkin and others. It is the latter ingredients which the mathematicians have developed and which is least familiar to physicists despite its great importance to the complete structure. My aim is therefore to explain it in a straightforward way which I hope is accessible to theoretical physicists with a knowledge of the theory of ordinary simple Lie algebras (and follows broadly the mathematical presentation of MACDONALD).

3 Summary of the theory of compact simple Lie algebras (of finite dimension).

If we choose an "orthonormal basis" the commutation relations of the Lie algebra g are :

$$[T^i, T^j] = i f^{ijk} T^k \qquad (3.1)$$

where the structure constants f^{ijk} are totally antisymmetric because we have chosen

$$Tr(T^i T^j) = \delta^{ij} \qquad (3.2)$$

where Tr denotes the invariant scalar product on g constructed by rescaling the trace of $T^i T^j$ evaluated in some finite dimensional irreducible representation of g.

The standard ploy for putting f^{ijk} into a more canonical form is to seek a Cartan subalgebra of mutually commuting generators (which cannot be extended) :

$$[H^i, H^j] = 0; \quad i,j = 1,.... \quad r = \text{rank } g . \qquad (3.3)$$

Complex linear combinations of the remaining generators are formed to constitute eigenstates under commutation with respect to the H^i :

$$[H^i, E^\alpha] = \alpha^i E^\alpha \quad . \qquad (3.4)$$

$\underline{\alpha}$ is a root and E^α its corresponding "step operator". The remaining commutators, between the step operators, must take the form :

$$[E^{\alpha}, E^{\beta}] = \begin{cases} \text{const } E^{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ \alpha.H & \text{if } \alpha+\beta = 0 \\ 0 & \text{if otherwise} \end{cases} \quad . \tag{3.5}$$

In this new basis, called the "Cartan Weyl" basis, the roots constitute structure constants with a remarkable crystallographic symmetry with respect to the "Weyl group" composed out of reflections in hyperplanes perpendicular to a root α :

$$\sigma_{\alpha}(x) = x - (2\alpha\cdot x/\alpha^2)\alpha \quad . \tag{3.6}$$

The roots can be split into two distinct sets called positive and negative such that if α and β are positive roots and $\alpha+\beta$ is a root then it is positive. Further any negative root is the negative of a positive root. This property is by no means obvious yet it can be achieved in a multitude of gauge equivalent ways. Given one such way we can further construct "simple roots" such that all the positive roots are obtained by adding them suitably. There are precisely r = rank g simple roots.

Their scalar products

$$K_{ij} = 2\alpha_{(i)}\cdot \alpha_{(j)}/(\alpha_{(j)})^2 \; (i,j = 1\ldots r). \tag{3.7}$$

have to be integers and form a r x r matrix called the Cartan matrix. Its diagonal elements are automatically equal to two so that only the off diagonal elements (which are zero or negative integers) carry non trivial information which can be encoded in the Dynkin diagram which consists of r points corresponding to the simple roots with points i and j joined by $K_{ij}\, K_{ji}$ lines.

The Dynkin diagram fully specifies the algebra since from it can be formed the simple roots and hence all the roots i.e. the structure constants in the Cartan Weyl basis.

In physics one requires finite dimensional irreducible representations of the Lie algebra g. Let $|\mu\rangle$ denote a basis state diagonalising the H^i in such a representation

$$H^i|\mu\rangle = \mu^i|\mu\rangle \tag{3.8}$$

We say μ^i is a weight and it has to satisfy

$$2\ \mu\cdot\alpha/\alpha^2 \ \varepsilon\ \mathbb{Z} \quad \text{any root } \alpha \text{ of } g \quad . \qquad (3.9)$$

Such a representation has a unique "highest weight" λ such that although λ is a weight $\lambda+\alpha$ never is if α is a positive root. λ satisfies (3.9) with the integer ≥ 0 if α is positive. The inequivalent irreducible unitary representations of g are classified by all such highest weights.

This summary is an "aide-memoire" and any reader unfamilar with the results is urged to consult a suitable text since henceforth a full understanding is assumed. I have written such lecture notes with Kac-Moody algebras in mind as a sequel (OLIVE). These notes are part of that intended sequel. See HUMPHREYS for a fuller treatment of Lie algebras.

4 (Untwisted) affine Kac-Moody algebras

Associated with the Lie algebra g (3.1) is the (untwisted) Kac-Moody algebra $\hat{g}$ whose generators T^i_n possess an integer valued suffix n in addition to the integer i, as in (3.1), and satisfy commutators :

$$[T^i_m,\ T^j_n] \ = \ i\ f^{ijk}T^k_{m+n} + k\ \delta^{ij} m\ \delta_{m+n,\ 0} \qquad (4.1a)$$

$$[k,\ T^i_m] \ = \ 0 \quad . \qquad (4.1b)$$

Notice that the integer suffix is "conserved" in (4.1a) suggesting to the physicist that it labels a sort of momentum. In fact roughly speaking (4.1) specifies a current algebra in momentum space including a Schwinger term with "c-number" coefficient k (SCHWINGER). T^i_o being the "current at zero momentum transfer" should be the charge and indeed satisfies (3.1). It can be shown that the "Schwinger term" satisfies the necessary Jacobi identities and is in a sense the most general c number term that can be included in the algebra $\hat{g}$ which is called a "loop algebra" if k = 0 (see GODDARD). Equations (4.1) appear to be a rather trivial extension of (3.1) but the occurrence of the Schwinger term affords a surprisingly rich structure as we shall see (and as was already appreciated by physicists in the '60's).

More precisely if we consider a quark model current algebra in two dimensions the light cone components of the currents, $j^i_t \pm j^i_x$ mutually commute and each satisfy (4.1) in momentum space if the light cone variables are compactified into circles so that momentum is quantised.

Other Kac-Moody algebras exist besides algebras (4.1). These are called "twisted" algebras but do not seem to have such ready physical applications and so will not be discussed specifically apart from a list of their Dynkin diagrams in section 6. They are essentially subalgebras of the untwisted algebras (4.1).

Let us now imitate the discussion of section 3 concerning finite dimensional Lie algebras and seek Cartan subalgebra generators and step operators for roots. Changing to the Cartan Weyl basis for g we have generators E^α_n, H^i_n and k and we see from (4.1) that H^i_o (i = 1....r) and k mutually commute. By (4.1)

$$[H^i_o, E^\alpha_n] = \alpha^i E^\alpha_n \tag{4.2a}$$

$$[k, E^\alpha_n] = 0 \quad . \tag{4.2b}$$

Thus the roots (α,o) appear to be infinitely degenerate since they possess corresponding step operators E^α_n for any n. We can "split the degeneracy" by introducing a new generator d, called the derivation satisfying

$$[d, T^i_n] = n\, T^i_n, \quad [d, k] = 0 \quad . \tag{4.3}$$

Physicists recognise d as an energy or momentum operator and it will equal minus the Virasoro generator L_0 of Goddard's lecture. By (4.3) d can be used to extend the dimension of the Cartan subalgebra by one to r+2. Also

$$[d, E^\alpha_n] = n\, E^\alpha_n \quad , \tag{4.4}$$

so that with respect to the Cartan subalgebra (H^i_o, k, d) we have by (4.2) and (4.4), step operators E^α_n with corresponding roots a :

$$E_n^{\alpha} \quad : \quad a \;=\; (\alpha,\, 0,\, n) \quad . \tag{4.5}$$

The remaining generators H_n^i constitute step operators for roots $n\ell$ where

$$H_n^i \quad : \quad n\ell \;=\; (0,\, 0,\, n) \tag{4.6}$$

Note that these roots differ from those in (4.5) by being r-fold degenerate in that the H_n^i act as step operators for any values of $i = 1,\ldots..r$. This is the first striking new feature compared to finite dimensional Lie algebra theory.

The root system of $\hat{g}$ is infinite of course yet spans a finite (r+1) dimensional space. Evidently we can define a set of positive roots by

$$(\alpha,\, 0,\, n), \quad (0,\, 0,\, n) \qquad n > 0$$

$$\text{and} \qquad (\alpha,\, 0,\, 0) \qquad \alpha > 0 \tag{4.7}$$

where $\alpha > 0$ is in the sense of g in section 3. Considering the subset of (4.7) with $n > 0$ we see that the old simple roots (of section 3) furnish simple roots of $\hat{g}$:

$$a_i \;=\; (\alpha_i,\, 0,\, 0) \qquad i = 1,\ldots..r \tag{4.8a}$$

We also need a simple root with $n = 1$ which has to be

$$a_o \;=\; (-\psi,\, 0,\, 1) \tag{4.8b}$$

where ψ is the highest root of g. In order to construct the Dynkin diagram for the algebra $\hat{g}$ we need to know the scalar product between the roots but for this we need the analogue of (3.2).

5 Invariant scalar product on $\hat{g}$

The representations of the Kac-Moody algebra $\hat{g}$, (4.1) with non zero c-number k are all infinite dimensional. Hence we cannot define an invariant scalar product of two generators in terms of a trace evaluated in some matrix representation as this definition would very likely diverge. We have to proceed less directly and the result will be the first big surprise, namely that the metric so found is indefinite with a single negative eigenvalue (and all others positive). Thus it resembles the metric of space time found by Einstein in his theory of relativity and will be called "Lorentzian". This resemblance will be reinforced by the connection with string theory and the physical applications there. The Lorentzian metric is such a crucial result that we shall establish it in detail here.

We shall notwithstanding denote the scalar product by Tr as in section 2. It must have the properties of (a) symmetry and (b) invariance :

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA) \tag{5.1a}$$

$$0 = \mathrm{Tr}\,([A, BC]) = \mathrm{Tr}\,([A,B]C) + \mathrm{Tr}\,(B[A,C]) \tag{5.1b}$$

where A, B and C are generators of $\hat{g}$. We know that within g such a product is essentially unique and given by (3.2). Hence

$$\mathrm{Tr}\,(T_o^i\, T_o^j) = \delta^{ij} \quad . \tag{5.2}$$

Taking $A = d$, $BC = T_m^i\, T_n^j$ in (5.1b) we find by (4.3) that

$$\mathrm{Tr}\,(T_m^i\, T_n^j) = 0 \quad \text{unless } m+n = 0 \quad . \tag{5.3}$$

Taking $A = T_m^i$, $BC = kT_n^j$ in (5.1b) we find by (4.1)

$$\mathrm{Tr}\,(k\, T_m^i) = 0 \tag{5.4}$$

$$\mathrm{Tr}\,(k^2) = 0 \tag{5.5}$$

Taking $A = T_{-n}^k$, $BC = T_o^i\, T_n^j$ (5.1b) reduces, using (5.2) and (5.4) to

$$\mathrm{Tr}\,(T_m^i\, T_{-m}^j) = \delta^{ij}$$

which combined with (5.3) can be rewritten as

$$\mathrm{Tr}\,(T^i_m\, T^j_{-n}) \quad = \quad \delta^{ij}\,\delta_{mn} \qquad . \tag{5.6}$$

Finally, taking $A = T^i_m$, $BC = d\,T^j_n$ we obtain from (5.1b)

$$\mathrm{Tr}\,(kd) \quad = \quad 1 \tag{5.7}$$

$$\mathrm{Tr}\,(dT^i_m) \quad = \quad 0 \qquad . \tag{5.8}$$

We see that by (5.4) and (5.8) that the two dimensional subspace of $\hat{g}$ spanned by k and d is orthogonal to the infinite dimensional subspace spanned by the T^i_m which is positive definite as we now show. The hermiticity properties of $\hat{g}$ are

$$T^{i\,\dagger}_n \; = \; T^i_{-n}, \qquad k^\dagger = k, \qquad d^\dagger = d \tag{5.9}$$

Hence the hermitian generators $(T^i_m + T^i_{-m})/\sqrt{2}$ and $(T^i_m - T^i_{-m})/\sqrt{2}i$ form an orthonormal set by (5.6).

By (5.5) and (5.7) the metric in the k,d subspace forms a matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \quad .$$

where $x = \mathrm{Tr}\, d^2$ is undetermined. The eigenvalues λ_1 and λ_2 of the matrix satisfy the secular equation

$$\lambda\,(\lambda - x) \;\; -1 \quad = \quad 0$$

so that their product $\lambda_1\lambda_2 = -1$ (whatever x). Hence one of these eigenvalues is negative irrespective of the undetermined quantity x which can actually be redefined to vanish as we now see. Note that the commutation relations (4.1), (4.3) are unchanged if we replace $d \to d' = d - xk/2$. After this redefinition we have $\mathrm{Tr}\, d^2 = 0$ which we shall henceforth assume.

The occurrence of this single negative eigenvalue is what we meant by saying the "metric" on $\hat{g}$ is "Lorentzian" and it will have an important influence on what follows.

We commented that the metric defined directly as a trace in some representation would very likely diverge yet we have succeeded in finding an algebraic way of ascribing a finite meaning to the metric. A physicist would call this a "renormalisation procedure".

6 Scalar products of roots of $\hat{g}$

The invariant metric we have just constructed defines by restriction a metric on the Cartan subalgebra of $\hat{g}$ formed by (H_0^i, k, d) which is also Lorentzian. This in turn enables us to define a product on possible eigenvalues of the Cartan subalgebra, $m = (\mu, \mu_k, \mu_d)$:

$$(m,m') = \underline{\mu}\cdot\underline{\mu}' + \mu_k\mu_d' + \mu_d\mu_k' \quad . \qquad (6.1)$$

Hence the roots of the form (4.5) have scalar product

$$(a,a') = \underline{\alpha}\cdot\underline{\alpha}' \qquad (6.2)$$

while those of the form (4.6) yield

$$(a,n\ell) = 0 \qquad (n\ell, n\ell) = 0 \quad . \qquad (6.3)$$

We see that the roots (4.5) and (4.6) are quite distinct in character. The roots of type (4.5) are called "real roots" and have positive length. The roots of type (4.6) are called "imaginary roots" and are orthogonal to all the roots including themselves. This is only possible because the metric (6.1) is Lorentzian.

In the language of relativity theory the imaginary roots are "light like". As we saw they are r-fold degenerate just as the photon which is the quantum of light propagates with a given light-like momentum in r possible linearly independent transverse polarisation states. This resemblance is not accidental. In the vertex operator construction the step operators for the imaginary roots are represented by vertices for the photons (or gauge particles) of the string theory.

It also follows from (6.2) and (6.3) that the Cartan matrix of $\hat{g}$ (formed of scalar products of its simple roots a_i) :

$$K_{ij} = 2(a_i, a_j)/(a_j, a_j) \quad i,j = 0,1,2,\ldots\ldots r \qquad (6.4)$$

equals the "extended Cartan matrix" of the finite dimensional Lie algebra g obtained from the ordinary Cartan matrix (3.7) by adding an extra row and column associated with the root $\alpha_o = -\psi$.

One way of calculating K_{oi} and K_{jo} is to observe that if λ_i are the "fundamental weights" of g, defined by

$$2\,\lambda_i \cdot \alpha_j/(\alpha_j)^2 = \delta_{ij} \quad i,j = 1,\ldots\ldots r \qquad (6.5)$$

(so that any weight of g, (3.9) is an integer linear combination of the λ_i and vice versa), then by (6.4) and (6.5)

$$\psi = -\sum_{i=1}^{r} K_{oi}\lambda_i \qquad (6.6)$$

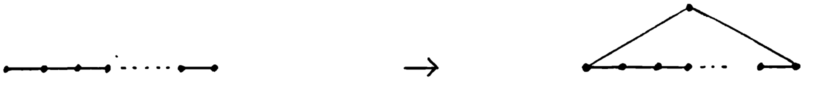
Further, as ψ is a long root of g

$$K_{io} = \begin{cases} 0 \text{ if } K_{oi} = 0 \\ -1 \text{ if } K_{oi} \neq 0 \end{cases} \qquad (6.7)$$

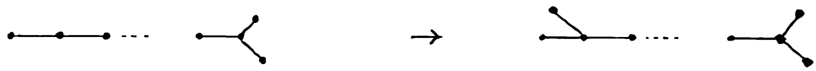
These observations enable us to construct the Dynkin diagram of $\hat{g}$ from that of g given (6.6). For example every particle physicist knows that the SU(3) octet is a quark antiquark bound state. More generally for SU(N)

$$\psi = \lambda_1 + \lambda_{N-1}$$

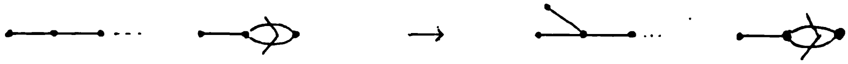
Hence the SU(N) Kac-Moody algebra Dynkin diagram is obtained from that for SU(N) by adding an extra point (0) and joining it to the two extremities of the SU(N) Dynkin diagram



Likewise $\psi = \lambda_2$ for the orthogonal group Lie algebras and the extra point (o) is joined to the penultimate point of the SO(N) Dynkin diagram

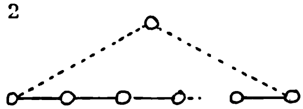


or

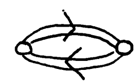


This procedure can be repeated to obtain from the Dynkin diagrams for the simple Lie algebras g the Dynkin diagrams for all the associated (untwisted) Kac-Moody algebras $\hat{g}$. The results are indicated below with the added lines drawn with dots

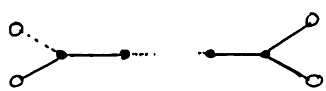
$\hat{A}_r$: r > 2



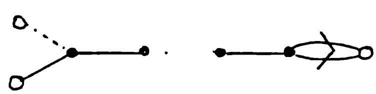
r=1



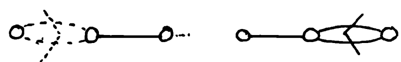
$\hat{D}_r$: r > 4



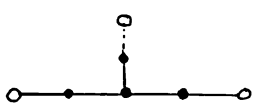
$\hat{B}_r$



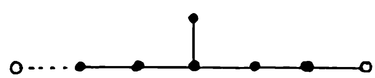
$\hat{C}_r$



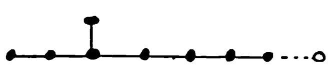
$\hat{E}_6$



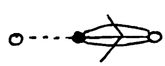
$\hat{E}_7$



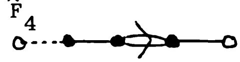
$\hat{E}_8$



$\hat{G}_2$



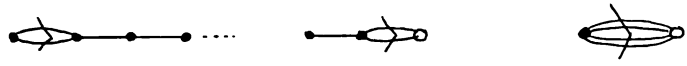
$\hat{F}_4$



Notice that the extended Dynkin diagram formed by adding the point (0) often has more symmetry than the original diagram and never less. The gain in symmetry can be related in a precise way to the centre of the simply connected group whose Lie algebra is g (OLIVE and TUROK 1983a). For example SU(r+1) has centre Z_{r+1} and this is the cyclic symmetry of the $\hat{A}_r$ Dynkin diagram (which has a 2 (r+1) element dihedral symmetry group). Likewise the group of E_6 has centre Z_3 which also occurs as a symmetry of the $\hat{E}_6$ Dynkin diagram (which altogether has S_3 symmetry).

We can also draw the Dynkin diagrams of the "twisted" Kac-Moody algebras. Apart from the last two these are obtained from the untwisted ones by interchanging long and short roots so that the arrows are reversed :

and



The significance of the distinction between solid (●) and open (O) vertices will be explained later.

7 The Weyl group of $\hat{g}$

The Weyl reflection in the hyperplane perpendicular to the real root a is specified by :

$$s_a(x) = x - [2(x,a)/(a,a)]\, a \qquad (7.1)$$

in analogy with (3.6) for the finite dimensional Lie algebra g except that the "Lorentzian" metric (6.1) must be used in (7.1) and that (7.1) is defined only for real roots and not imaginary roots as (a,a) vanishes for them. The Weyl group $\hat{W}$ is the infinite discrete group generated by the reflections in the real roots of $\hat{g}$. As ℓ the primitive imaginary root is perpendicular to any real root of $\hat{g}$ by (6.3) we have

$$s_a(\ell) = \ell \qquad (7.2)$$

for any real root a. Hence each imaginary root is left invariant under the action of any element of $\hat{W}$. On the other hand real roots are reflected into real roots so that the set of real roots is permuted by $\hat{W}$.

We shall now show that $\hat{W}$ has the general structure

$$\hat{W} = W \alpha \Lambda_r v \qquad (7.3)$$

that is it is isomorphic to the semidirect product of the finite group W (the Weyl group of g) with $\Lambda_r v$, the coroot lattice of g. ($\Lambda_r v$ is the lattice spanned by the coroots $\alpha^{v} = \alpha/\alpha^2$ where α are roots of g). This is rather like the statement that the Poincare group is isomorphic to the semidirect product of the Lorentz group with the translation group.

To see (7.3) we let s_a (7.1) act on

$$x = (\xi, \kappa, \delta) \quad . \tag{7.4}$$

By equations (4.5), (3.6), (6.1), (7.1) and (7.4)

$$s_a(x) = (\sigma_\alpha(\xi + 2n\kappa\alpha/\alpha^2), \kappa, \delta + [\xi^2 - (\xi + 2n\kappa\alpha/\alpha^2)^2]/2\kappa)$$

Hence

$$s_a = s_{\alpha+n\ell} = \sigma_\alpha(t_{\alpha^v})^n$$

where t_{α^v} can be thought of as a displacement by 2κ times the coroot α^v and

$$t_{\alpha^v}(x) = (\xi + 2\kappa\alpha/\alpha^2, \kappa, \delta + [\xi^2 - (\xi + 2\kappa\alpha/\alpha^2)^2]/2\kappa)$$

It is easy to check that

$$\sigma_\beta \, t_{\alpha^v} \, \sigma_\beta = t_{\sigma_\beta(\alpha^v)}$$

from which it follows that $\Lambda_r v$ is an invariant subgroup of $\hat{W}$ and that $\hat{W}$ has the structure (7.3). It is easy to visualise $\hat{W}$ as a group of invariances of $\Lambda_r v$. $\Lambda_r v$ itself furnishes the translation invariances while the subgroup of $\hat{W}$ which leaves a point of $\Lambda_r v$ fixed is isomorphic to W. It is an interesting historical fact that groups of this type were classified by Coxeter using a Dynkin diagram notation long before the recent developments (COXETER p.194).

8 Highest Weight Representations of $\hat{g}$

Finite dimensional representations of the finite dimensional Lie algebra g have both highest and lowest weight states. These states are unique if the representation is irreducible.

In physical applications of Kac-Moody algebras $\hat{g}$, (-d) is usually identified physically with an operator such as the energy or scale operator whose spectrum must be bounded below. It follows that the representation considered must possess highest weight states. Then as shown below the c-number k must be positive. Then there must be no lowest weight states or else k would also be negative, which is a contradiction. Hence the representations are infinite dimensional.

The analysis of possible highest weight states of $\hat{g}$ we now undertake superficially resembles the usual treatment for g. Let $\ell = (\lambda, \kappa, \delta)$ be a highest weight in the sense that while it is a weight of the representation considered $(\ell+a)$ never is if a is a positive root of $\hat{g}$. If $|\ell\rangle$ is the corresponding state we have

$$E_a|\ell\rangle = 0 \qquad a > 0 \tag{8.1}$$

$$H_0^i|\ell\rangle = \lambda_i|\ell\rangle, \; k|\ell\rangle = k|\ell\rangle, \; d|\ell\rangle = \delta|\ell\rangle \quad . \tag{8.2}$$

We consider the sequence of all possible states obtained from $|\ell\rangle$ by acting with step operators for negative roots

$$|\ell\rangle, \; E_{-a}|\ell\rangle, \; E_{-b}E_{-a}|\ell\rangle, \; E_{-c}E_{-b}E_{-a}|\ell\rangle, \ldots. \tag{8.3}$$

(with a, b, c.... positive roots). Then these states have weights

$$\ell, \; \ell-a, \; \ell-a-b, \; \ell-a-b-c, \ldots. \tag{8.4}$$

respectively. It is possible to show that no new states are obtained by acting on the states (8.3) with step operators for positive roots. Hence the set (8.3) span an subspace invariant under $\hat{g}$ with the feature that the possible weights m all differ from ℓ by a sum of positive roots. We write

$$m < \ell \quad . \tag{8.5}$$

Thus ℓ is in fact the unique highest weight in the representation defined by the spaces spanned by (8.3) and can be used as a label.

Corresponding to the Weyl reflection (7.1) in the real root a there exists a unitary element S_a of the SU(2) group obtained by exponentiating the SU(2) subalgebra associated with the real root a such that $S_a|\ell\rangle$ has weight $s_a(\ell)$. Since this weight must belong to the set (8.4) we have, by (8.5), $s_a(\ell)<\ell$ which implies, by (7.1)

$$2(a,\ell)/(a,a) = \text{integer} \geq 0$$
$$\text{for each positive real root a.} \tag{8.6}$$

We can solve (8.6) just as in the g analysis by defining r + 1 fundamental weights ℓ_i of $\hat{g}$ satisfying

$$2(\ell_i, a_j)/(a_j, a_j) = \delta_{ij} \qquad i,j = 0, 1, \ldots\ldots r \qquad (8.7)$$

where a_i are the (r+1) simple roots (4.8). Then the general solution to (8.6) is

$$\ell = \sum_{i=0}^{r} x_i \ell_i, \qquad x_i \text{ an integer} \geq 0 \qquad (8.8)$$

apart from an indeterminate component in the d direction.

Using the definition (6.5) for the fundamental weights λ_i of g and (4.8) we find that the fundamental weights ℓ_i of $\hat{g}$, (8.7) are

$$\ell_i = (\lambda_i, m_i \psi^2/2, o) \quad i = 1, \ldots . r \qquad (8.9a)$$

$$\ell_o = (0, m_o \psi^2/2, 0) \qquad (8.9b)$$

where the last entry, (the d component) is undetermined and put equal to zero arbitrarily. The coefficients m_i are integers ≥ 1 defined by

$$\psi/\psi^2 = \sum_{i=1}^{r} m_i \alpha_i/(\alpha_i)^2, \qquad m_o = 1 \qquad (8.10)$$

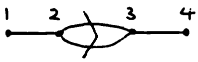
The $2k/\psi^2$ component of ℓ has a special structure and is called the "level" of ℓ. By (8.8) and (8.9) it is given by

$$\text{level } \ell = \sum_{i=0}^{r} m_i x_i \qquad (8.11)$$

It is a positive integer. Thus $2k/\psi^2$ is quantised and positive in a highest weight representation.

The level 1 or "basic" representations are particularly significant since higher level representations can be built up by considering direct products of basic representations and reducing into invariant subspaces. We shall now enumerate these basic representations. As $m_i \geq 1$ expression (8.11) can equal unity only if ℓ is a fundamental weight ℓ_i and if, in addition $m_i = 1$.

Books on Lie algebra usually tabulate the expansion of the highest root ϕ in terms of simple roots α_i (see, for example HUMPHREYS p.66). Thus for F_4

F_4 : 

$$\phi = 2\alpha_1 + 3\,\alpha_2 + 4\,\alpha_3 + 2\,\alpha_4$$

Hence

$$\phi/\phi^2 = 2\,\alpha_1/(\alpha_1)^2 + 3\,\alpha_2/(\alpha_2)^2 + 2\,\alpha_3/(\alpha_3)^2 + \alpha_4/(\alpha_4)^2\ .$$

Thus for $\hat{F}_4$ only m_o and m_4 equal 1 and the only basic weights are ℓ_0 and ℓ_4. The corresponding vertices are denoted as open (rather than closed) dots in the Dynkin diagrams of section 6 where the results of the corresponding calculations for all the Kac Moody algebras $\hat{g}$ are indicated.

Some of these results can be obtained without calculation. For example $m_0 = 1$ always by (8.10). The extra symmetry of the extended Dynkin diagrams that we commented on becomes important now since any point (i) symmetrically related to the point (0) must also have $m_i = 1$. Thus all points of the $\hat{A}_r$ Dynkin diagram are symmetrically related to point (0) and hence all $m_i = 1$. Similarly for $\hat{D}_r$ all m_i for the four tip points equal 1. For the $\hat{A}_r$, $\hat{D}_r$, $\hat{E}_r$ Kac-Moody algebras whose Dynkin diagrams are simply laced all points with $m_i = 1$ are obtained in this way as being symmetrically related to the point (0).

Representations of ordinary Lie algebras g whose highest weights are related by a symmetry of the Dynkin diagram of g are isomorphic (though not equivalent) because of the existence of an outer automorphism of g. For example D_4 (SO(8)) has three isomorphic 8 dimensional representations, the vector and the two spinor representations, associated with the three tips of its Dynkin diagram  . Presumably a similar statement holds for the representations of Kac-Moody algebras $\hat{g}$ whose highest weights are likewise related by symmetries of the Dynkin diagram of $\hat{g}$. If so, the basic representations of the simply laced Kac-Moody algebras ($\hat{A}_r$, $\hat{D}_r$ and $\hat{E}_r$) are all isomorphic so that in some sense each of these algebras has only one basic representation.

Finally let us point out that the coefficients m_i defined by (8.10) and occurring in (8.11) can be related directly to the

Cartan matrices (6.4) and hence the Dynkin diagram of $\hat{g}$. They satisfy

$$\sum_{j=0}^{r} K_{ij} m_j = 0, \quad i = 0 \ldots r \quad ; \quad m_j \text{ positive integers with no common factor.} \qquad (8.12)$$

Thus the m_j constitute the right null vector of the Cartan matrix. This characterisation can also be applied to the twisted algebras whose Dynkin diagrams were listed at the end of section 6.

9 Representation theory and quantum fields

The representation theory as developed so far has seemed to be a straightforward extension of that for ordinary finite dimensional Lie algebras g except that the tendency of Dynkin diagrams of Kac Moody algebras to have greater symmetry has meant that more representations are isomorphic.

One might have thought that because the physically interesting representations are infinite dimensional they would be intractable. This is not so. Characters can be defined and a theory developed in analogy with the ordinary case leading to new perspectives on a variety of combinatorial identities; Rogers-Ramanujan identities, MacDonald identities, and so on. What is even more interesting is that the methods of quantum field theory furnish means of constructing representations of Kac-Moody algebras which have no parallel for finite dimensional algebras. The quantum fields concerned may be either bosonic or fermionic in character. They depend on a complex variable z and are analytic in the neighbourhood of the unit circle. The coefficients of z^{-n} in the Laurent expansion of these fields constitute bosonic or fermionic oscillators respectively.

The vertex operator construction of dual string theory involves an r component bose field $Q^i(z)$, called the Fubini Veneziano field and represents H_n^i and E_n^α by the coefficients of z^{-n} in the Laurent expansions of $iz \frac{dQ^i}{dz}$ and $z : e^{i\alpha \cdot Q(z)}:$ respectively. This construction yields all the "basic" or level 1 representations of the $\hat{A}_r$, $\hat{D}_r$ and $\hat{E}_r$ Kac-Moody algebras when the real roots are normalised by $a^2 = 2$ and when a certain sign correction factor is included (FRENKEL and KAC). Thus the space generated from the highest weight (8.3) is seen to be made up from the Fock spaces of the oscillators. A more complicated construction of this type has been found for the remaining, non-simply laced Kac-Moody algebras (FEINGOLD and FRENKEL). In checking the construction it is crucial to take very careful account of normal ordering (denoted by double dots above) which is specifically a quantum field theory phenomenon.

The same care must be excercised with the second construction well known in particle physics as the quark model for current algebras, in order to obtain the c number k. A real finite (d) dimensional representation of g, (3.1), t^i, say, is considered. d real fermion fields $H^\alpha(z)$ are defined on the unit circle of the z plane to be either periodic (RAMOND) or antiperiodic; (NEVEU and SCHWARZ). Then the T^i_n satisfying (4.1a) are given by the coefficient of z^{-n} in the Laurent expansion of $H^\alpha(z)t^i_{\alpha\beta}H^\beta(z)/2$. The c number is then given by $(1/2)\ \mathrm{Tr}(t^i t^j) = k\ \delta^{ij}$, which is essentially the Dynkin index of the representation t. Although such representations evidently have positive k they are not irreducible in general and some work is needed to calculate the highest weights and their possible degeneracy. For example the defining representations of the classical Lie algebras yield the smallest possible value of $2k/\psi^2$, namely 1. Hence the quark construction then reduces to basic representations but only for the orthogonal series is the number of irreducible components finite. In the D_r case two of the basic representations are found if the Ramond fields are used and the other two if the Neveu-Schwarz fields are used.

This is an interesting result since the same representations were found by the vertex operator constructions above and hence it must be possible (with care) to equate the two expressions for the generators, one involving boson fields and one involving fermi fields. These identities are indeed valid and can be checked explicitly and constitute part of Skyrme's fermion boson equivalence mentioned in the introduction; (SKYRME). The remaining part is the expression of the fermion field as a vertex operator. It is amazing that Skyrme's results predated both the current algebra and vertex operator constructions in physics.

The point to be made here (FRENKEL 1981) is that the validity of the fermion-boson equivalence can be anticipated from the classification of representations of Kac-Moody algebras by highest weights explained in section 8. In all likelihood there will be other such identities corresponding to other, yet to be explored, quantum field theoretic constructions.

Another development concerns the introduction of (r+2) component "covariant" bose fields instead of the r component "transverse" bose fields used above. The vertex operator construction can be extended to "Lorentzian algebras" in which the simply laced Kac-Moody algebras naturally sit. Lorentzian algebras corresponding to self dual even lattices promise to be particularly interesting both for mathematics and physics (FRENKEL 1982; GODDARD and OLIVE 1983).

It is also possible to construct a "Virasoro algebra" with generators L_n such that $L_o+d = 0$ out of a given highest weight representation of a Kac-Moody algebra using a natural quantum field theory construction (SUGAWARA; SOMMERFIELD). This leads to further constructions (GODDARD and OLIVE 1984; GODDARD, KENT and OLIVE) which seem to be relevant to the theory of critical phenomena in two dimensional lattice models such as the Ising model and generalisations: (FRIEDAN, QIU and SHENKER).

The aim of this last section (9) has been to briefly review what I said in my lectures and to provide an introduction to some of the other developments of the theory. No doubt many more are to come.

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REFERENCES

ADLER S. and DASHEN R. : Current Algebras and applications to particle physics (Benjamin, New York 1968).

COXETER H.S.M. : Regular Polytopes : (1st Edition 1947; 3rd Edition Dover 1973).

FEINGOLD A.J. and FRENKEL I. : "Classical Affine Algebras" (preprint).

FRENKEL I. (1981) : Journal of Functional Analysis 44/1981/259-327.

FRENKEL I. (1982) : "Representations of Kac-Moody Algebras and dual resonance models" (preprint).

FRENKEL I. and KAC V.G. : Inventiones Math. 62/1980/23-66.

FRIEDAN D., QIU Z. and SHENKER S. : "Conformal Invariance, Unitarity and two dimensional critical exponents" EFI 83-66 Preprint (1983); Phys. Rev. Lett. 52/1984/1575-1578.

FUBINI S., GORDON D. and VENEZIANO G. : Phys. Lett. 29B/1969/679-682.

FUBINI S. and VENEZIANO G. : Nuovo Cim. 67A/1970/29-47.

GELLMANN M. : Phys. Rev. 125/1962/1067-1084.

GELLMANN M. and NE'EMAN Y. : The Eightfold Way (Benjamin New York 1964).

GODDARD P. : Lectures at this meeting.

GODDARD P., KENT A. and OLIVE D. : "Virasoro Algebras and Coset Space Models" DAMTP preprint 84-22.

GODDARD P. and OLIVE D. (1983) : "Algebras, Lattices and Strings", DAMTP preprint 83-22.

GODDARD P. and OLIVE D. (1984) : "Kac-Moody Algebras, Conformal Symmetry and Critical Exponents", DAMTP preprint 84-16.

GREEN M.B. and SCHWARZ J. : Phys. Lett. 149B/1984/117-122.

GROSS D., HARVEY J., MARTINEC E. and ROHM R. : "The Heterotic String", Princeton preprint 1984.

HUMPHREYS J.E. : Introduction to Lie Algebras and Representation Theory (Springer, Berlin 1972).

JACOB M. : Dual Theory (North Holland, Amsterdam 1974).

JIMBO M. and MIWA T. : Publ. RIMS, Kyoto Univ. 19/1983/943-1001.

KAC V.G. (1968) : Matt. USSR-Izv.2/1968/1271-1311.

KAC V.G. (1983) : Infinite Dimensional Lie Algebras (Birkhauser, Boston 1983).

MACDONALD I.G. : Springer Lecture Notes in Maths 901/1981/258-276.

MOODY R. : Journ. Alg. 10/1968/211-230.

NAMBU Y. : Proc. Intern. Conf. on Symmetries and Quark Models, Wayne University, 1969 (Gordon and Breach, New York 1970), 269.

NEVEU A. and SCHWARZ J. : Nucl. Phys. B31/1971/86-112.

OLIVE D. : Lectures on Gauge Theories and Lie Algebras with some applications to spontaneous symmetry breaking and integrable dynamical systems. UVA preprint 1983.

OLIVE D. and TUROK N. (1983a) : Nucl. Phys. B215 [FS7]/1983/470-494.

OLIVE D. and TUROK N. (1983b) : Nucl. Phys. B220 [FS8]/1983/491-507.

OLIVE D. and TUROK N. (1984) "Local conserved densities and zero curvature conditions for Toda lattice field theories". Imperial preprint TP/84-85/5.

RAMOND P. : Phys. Rev. D3/1971/2415-2418.
SCHWINGER J. : Phys. Rev. Lett. 3/1959/296-297.
SKYRME T.H.R. : Proc. Roy. Soc. A262/1961/237-245.
SOMMERFIELD C.M. : Phys. Rev. 176/1968/2019-2025.
SUGAWARA H. : Phys. Rev. 170/1968/1659-1662.

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