A. K. Kwaśniewski

Critical curves in nonstandard Potts models

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 9. pp. [137]--139.

Persistent URL: http://dml.cz/dmlcz/701395

# CRITICAL CURVES IN NONSTANDARD POTTS MODELS

A.K. Kwaśniewski

In 1952, using a Kramers-Wannier type analysis, Potts has reported [6] critical points for the spin system on the lattice with Kronecker $\delta$-like interaction of the nearest neighbours.
The other, cosine-like interaction, generalizing the Ising model one was also considered there, however no similar results were obtained.
This is in this very case - which we call the nonstandard (or planar [1]) Potts model - that we derive equations of critical curves with the method of Kramers and Wannier due to a generalization [2] of Onsager-Kaufman description of the Ising system via Clifford algebras.

Consider then the system on the two-dimensional torus lattice with p rows and q columns. Let its state be described by a p×q matrix $(s_{ik})$, $s_{ik} \in Z_n$.

We denote by $Z_n$ the multiplicative cyclic group of n-th roots of unity while $Z_n'$ stands for its additive realization.

The total energy of the system in a given state, in the case of nonstandard Potts model reads as follows:

$$-\frac{E[(s_{ik})]}{kT} = a \sum_{i,k=1}^{p,q} \left( s_{ik}^{-1} s_{i,k+1} + s_{i,k+1}^{-1} s_{ik} \right) + b \sum_{i,k=1}^{p,q} \left( s_{ik}^{-1} s_{i+1,k} + s_{i+1,k}^{-1} s_{ik} \right) . \tag{1}$$

The transfer matrix M for this model can be represented in a convenient form [2] with use of generalized Pauli matrices [4] and generalized "cosh" functions $f_i$ , $i \in Z_n'$ [2]:

$$f_i(x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-ki} \exp\{\omega^k x\} \quad , \tag{2}$$

where x might be any element of some associative algebra with unity while $\omega$ stands for the generator of $Z_n$.
The crucial property of $f_i$'s is their relation to the eigenvalues $\chi_k$ of the interaction matrix W [2,3] i.e.

$$\sum_{i=0}^{n-1} f_i(a)\, f_{i-k}(a) = \frac{1}{n}\, \chi_k a\ , \qquad k\in Z_n' \quad , \tag{3}$$

where the known $\chi_k(a)$ , $k\in Z_n'$ (with $\chi_k=\chi_{-k}$) form the set of all eigenvalues of the circulant matrix W:

$$W = \sum_{l=0}^{n-1} \exp\{2a\ \mathrm{Re}\omega^l\}\sigma_1^l = W[a]. \tag{4}$$

Here $\sigma_1=(\sigma_{i+1,j})$, $i,j\in Z_n'$, denotes one of the three $n\times n$ $\sigma_1,\sigma_2,\sigma_3$, generalized Pauli matrices which are defined to satisfy the following relations:

$$\sigma_i\sigma_j = \omega\sigma_j\sigma_i \qquad i<j, \qquad \sigma_i^n = I\ ; \quad \sigma_i=(n\times n)$$

[4,2,3].

As in the Ising model case, introducing the tensor products

$$X_k = I\otimes\ldots\otimes I\otimes\sigma_1\otimes I\otimes\ldots\otimes I \qquad \text{(p terms)} \tag{5}$$

$$Z_k = I\otimes\ldots\otimes I\otimes\sigma_3\otimes I\otimes\ldots\otimes I \qquad \text{(p terms)} \tag{6}$$

- where Pauli matrices are placed at the k-th site - one arrives at the following form of the transfer matrix [2,3]

$$M = [g(a^*)]^p \exp\left\{b\sum_{k=1}^{p} Z_k^{-1}Z_{k+1} + Z_{k+1}^{-1}Z_k\right\} \exp\left\{a^*\sum_{k=1}^{p}\left(X_k + X_k^{-1}\right)\right\} \tag{7}$$

where $[g(a^*)]^n=\det W[a]$ and $a^*$ is the dual parameter to be found from its defining relation:

$$\det W[a^*] = n^n \det W^{-1}[a] \quad . \tag{8}$$

We do not quote the boundary cyclic conditions as finally we are concerned with thermodynamic limit only.

There, the planar models under consideration posses the Kramers-Wannier duality property.

For that to demonstrate let us introduce the operators

$$\prod_{r<k} X_r = Z_k \qquad \text{and } Z_k^{-1}Z_{k+1} = X_k, \qquad k=1,\ldots,p. \tag{9}$$

Note then that these very operators do satisfy the same, generalized Clifford algebra defining relations as $X_k$ and $Z_k$. Hence (9) defines also an automorphism of the very algebra and this must be an inner automorphism because the generalized

Clifford algebra with 2p generators is isomorphic to the algebra of all $n^p \times n^p$ matrices [5].
This in turn means that there exists an invertible matrix D such that

$$DZ_kD^{-1} = Z_k \qquad \text{and} \qquad DX_kD^{-1} = X_k \quad . \tag{10}$$

At the same time, from (7) and (9) one gets

$$D^{-1}MD = [g(a^*)]^p \exp\left\{b \sum_k \left(X_k + X_k^{-1}\right)\right\} \exp\left\{a \sum_k \left(Z_k^{-1}Z_{k+1} + Z_{k+1}^{-1}Z_k\right)\right\} \quad . \tag{11}$$

Considering now the parameter b as the dual of the dual b* one arrives [3] at the following duality relation for the free energy F of the system:

$$F(a,b) = -\frac{1}{n} \ln \frac{\det W[a] \cdot \det W[b]}{n^n} + F(b^*,a^*) \tag{12}$$

Using now arguments similar to those of Kramers and Wannier, under the assumption of uniqueness of the existing critical curves, we conclude that their equation, in parameter's a and b plane, is of the form:

$$\det W[a] \ \det W[b] = n^n \quad . \tag{13}$$

The interaction matrix is easily diagonalizable and det W is known. One also readily verifies that for n=2, equation (13) becomes the one known in the Ising model case.

The author expresses his thanks to Z.Strycharski, M.Dudek and J.Lukierski.

REFERENCES

[1] DOMB C. J.Phys. A7, (1974), 1335
[2] KWASNIEWSKI A.K. Wroclaw Univ. preprint No.621 (Nov.) 1984
[3] KWASNIEWSKI A.K. Wroclaw Univ. preprint No.626 (Nov.) 1984
[4] MORRIS A.O. Quart.J.Math.Oxford (2) 18 (1967),7-12
[5] POPOVICI et al. C.R.Acad.Sc.Paris, t.262 (1966), 682
[6] POTTS R.B. Proc.Cam.Phil.Soc.48 (1952), 106

INSTITUTE OF THEORETICAL PHYSICS, UNIVERSITY OF WROCLAW, 50-205 WROCLAW, CYBULSKIEGO 36, POLAND