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UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

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In this talk I should like to describe recent work by Peter Goddard, David Olive and myself. Our results are reported in Goddard-Kent-Olive (1985); they develop ideas of Goddard-Olive (1984). Mathematicians can regard the work as a contribution to the representation theory of the Lie algebra with generators $\{L_m : m \in \mathbb{Z}\} \cup \{c\}$ over $\mathbb{C}$ and commutation relations:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{1}{12}\, cm(m^2-1)\,\delta_{m,-n} \qquad (1a)$$

$$[L_m, c] = 0\ . \qquad (1b)$$

This is the Virasoro algebra, the essentially unique central extension of the algebra of vector fields on the circle (the Witt algebra). Its first occurrence in physics was in Virasoro (1970), in the context of dual string theory (the central extension was previously considered in Gelfand-Fuks (1968)).

Recently, it has been utilized to study conformally invariant field theories and lattice systems in two dimensions: the conformal invariance implies that the states of these models must provide representations of the Virasoro algebra. By the nature of physical theories, in which the states belong to a Fock space built up from a ground state or 'vacuum', these representations must be highest weight representations, containing a highest weight vector $|h\rangle$ such that:

$$L_n|h\rangle = 0 \quad (n > 0) \qquad (2a)$$

$$L_0|h\rangle = h|h\rangle \qquad (2b)$$

The irreducible highest weight representation spaces are built up from $|h\rangle$ by repeated application of L_{-n}'s $(n > 0)$, and are thus completely specified by the parameters c,h of equations (1) and (2).

There are physical reasons for further restricting these representations. One condition, given by Belavin-Polyakov-Zamolodchikov (1984), is that the representation should be 'degenerate' (this term will be explained shortly), to allow the associated physical theory to be exactly solvable. An additional criterion, investigated by Friedan-Qiu-Shenker (1983), is that of unitarity, requiring:

$$(L_n)^+ = L_{-n} \tag{3}$$

with respect to a positive inner production on the state space.

A powerful tool for investigating the properties of the highest weight states is the formula of Kac (1979):

$$\det M_{(n)}(h,c) = C \prod_{pq\leq n} (h - h_{p,q}(c))^{P(n-pq)} \tag{4a}$$

where

$$c = 1 - \frac{6}{m(m+1)} \tag{4b}$$

$$h_{p,q}(c) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} \tag{4c}$$

Here $M_{(n)}(h,c)$ is the matrix of inner products of the $p(n)$ vectors $\{L_{-r_1}\ldots L_{-r_m}|h\rangle : \{r_1,\ldots,r_m\}$ is a partition of $n\}$ which span the space of level n vectors (those whose L_0 eigenvalue is $h+n$). The inner products are calculated by:

$$(L_{-r_1}\ldots L_{-r_m}|h\rangle, L_{-s_1}\ldots L_{-s_t}|h\rangle) = \langle h|L_{r_m}\ldots L_{r_1}L_{-s_1}\ldots L_{-s_t}|h\rangle \tag{5a}$$

with

$$\langle h|L_{-n} = 0 \quad (n > 0) \tag{5b}$$

$$\langle h|h\rangle = 1$$

making use of equations (1),(2), (hence their h and c dependence). C in equation (4a) is a non-zero constant.

The degenerate representations are those for which

$$h = h_{p,q}(c) \tag{6}$$

for infinitely many integers p,q. By careful analysis of the Kac formula, Friedan-Qiu-Shenker (1983) managed to obtain a condition for unitarity, constraining the representations available in quantum field theory (in which unitarity is required) or in an important class of lattice systems (those which are reflection positive: this implies unitarity). Their condition was <u>either</u> that $c \geq 1$, $h \geq 0$ <u>or</u> that (h,c) should obey equations (4) with (p,q,m) positive integers such that

$$1 \leq q \leq p < m \tag{7}$$

In the former case, the representations are guaranteed to be unitary. For $c < 1$, they showed the latter condition was necessary, but not that the representations were in fact unitary. They provided, however, strong physical reasons for believing that at least the first four sets of representations (corresponding to $m = 3,4,5,6$) were indeed unitary. The conformal invariance of the theory is reflected in the power law scaling behaviour of its correlation functions, regulated by the critical exponents of the model. For a given model, these exponents can be calculated (or obtained experimentally, if the model is realised in nature). If the model provides particular representations of the Virasoro algebra, the exponents are given by the values of h of these representations. FQS showed that the known exponents of the Ising, tricritical Ising, 3-state Potts and tricritical 3-state Potts models correspond to those predicted by the h-values obtained from (4c), (6), (7) for $m = 3,4,5,6$ respectively. These models are reflection positive.

There is thus both mathematical and physical motivation for answering the questions: Do unitary representations exist for the FQS series given by (4),(6),(7)? What is the structure of such representations? Physically, one might hope to find models corresponding to the rest of the series, and to learn something about the structure of both the old and the new models, provided the first mathematical question has an affirmative answer. As we shall see, it does.

To show this, we explicitly construct representations with all the values of c given by (4b), obeying the unitarity condition (3) with respect to the inner product on a Fock space (which is, of course, positive definite). The Fock space is that of the "quark model", the theory of a d-dimensional real fermion field $H^{\alpha}(z)$ $(1 \leq \alpha \leq d_{\lambda}\ ,\ |z| = 1)$ defined on the unit circle in the complex plane. There are two possible cases, considered by Ramond (1971) and Neveu-Schwarz (1971). In these, the fields have a Laurent expansion

$$H^{\alpha}(z) = \sum b^{\alpha}_{r}\, z^{-r} \tag{8}$$

with summation over $r \in \mathbb{Z}$ (R) or $r \in \mathbb{Z} + \frac{1}{2}$ (NS). The b^{α}_{r} are creation or annihilation operators with the anticommutation relations and unitarity condition:

$$[b^{\alpha}_{r}, b^{\beta}_{s}] = \delta^{\alpha\beta}\delta_{r,-s} \tag{9}$$
$$(b^{\alpha}_{r})^{+} = b^{\alpha}_{-r}$$

They define a fermionic Fock space built up by the b^{α}_{-r} $(r > 0)$ from a vacuum $|0\rangle$ (which in the Ramond case is degenerate) that is annihilated by the b^{α}_{r} $(r > 0)$.

In this theory we define, following Kac-Peterson (1983) a representation of the affine Kac-Moody algebra $\hat{g}$ associated to a Lie algebra g as follows. Consider a real matrix representation λ of g, by matrices $(M^{a})_{ij}$ $(1 \leq a \leq \dim g\ ,\ 1 \leq i,j \leq d_{\lambda})$. Set $d = d_{\lambda}$, and define the current

$$\sum_{n} T^{a}_{n}\, z^{-n} = J^{a}(z) = :H^{i}(z)\, M^{a}_{ij}\, H^{j}(z): \ , \tag{11}$$

$$:b^{\alpha}_{r}\, b^{\beta}_{-r}: = b^{\beta}_{-r}\, b^{\alpha}_{r} + \frac{1}{2}\,\delta^{\alpha\beta}\delta_{r,0} \qquad (r > 0)\ , \tag{12a}$$

$$:b^{\alpha}_{r}\, b^{\beta}_{s}: = b^{\alpha}_{r}\, b^{\beta}_{s} \qquad \text{(otherwise)}\ . \tag{12b}$$

The Laurent coefficients T^{a}_{n} $(n \in \mathbb{Z})$ obey:

$$[T^{a}_{m}, T^{b}_{n}] = if^{abc}T^{c}_{m+n} + \frac{k_{\lambda}}{2}\, m\delta^{ab}\delta_{m,-n} \tag{13}$$

where f^{ijk} and k_{λ} are the structure constants and Dynkin index of the representation λ , i.e.

$$[M^{a}, M^{b}] = f^{ab}{}_{c}M^{c} \tag{14a}$$

$$\mathrm{Tr}(M^a M^b) = -k_\lambda \delta^{ab} . \tag{14b}$$

Now if we define

$$\sum L_n^g z^{-n} = \frac{2}{c_\psi + k_\lambda^g} \,{}^\circ_\circ J^a(z)\; J^a(z)\,{}^\circ_\circ \tag{15}$$

we see that the coefficients L_n^g obey the equation (1a) with

$$c^g = \frac{k_\lambda^g d_\psi^g}{c_\psi^g + k_\lambda^g} , \tag{16}$$

where d_ψ^g, c_ψ^g are the dimension and quadratic Casimir opertors of the adjoint representation ψ of the algebra g. (Physicists will recognise (15) as a rescaling of a component of the energy-momentum tensor, bilinear in the currents, which generates the action of the conformal algebra on the fields in a conformally invariant, two-Euclidean-dimensional theory.) The reality of the representation λ guarantees the unitary equation (3).

Now all the values of c obtained by equation (16) for all possible algebras g and representations λ are larger than one. However, there is a further substructure to the constructions (11) and (15). Take a subalgebra h of g, which is represented by a subset of the matrices M^a. These can be chosen to be orthogonal in the sense of (14b). The argument of equations (11) to (16) applies equally to h, so that we obtain Virasoro generators L_n^h obeying (1a) with

$$c^h = \frac{k_\lambda^h d_\psi^h}{c_\psi^h + k_\lambda^h} . \tag{17}$$

Moreover, if we define

$$K_n = L_n^g - L_n^h \tag{18}$$

we find

$$[K_n, L_n^g] = 0 \tag{19}$$

and thus that the K_n themselves obey (1a) with

$$c = c^g - c^h . \tag{20}$$

In particular, take g and h to be $SU(2) \oplus SU(2)$ and the diagonal

SU(2). Take λ to be the direct sum of n copies of (the reali version of the spin($\frac{1}{2}$) representation of the first factor of g and 1 copy of the same representation of the second factor. The resulting values of c are precisely those of the series sought by FQS. This construction provides a highly reducible representation of the Virasoro algebra (18), and we expect that a careful analysis will show that highest weight vectors of all possible highest weights h corresponding to each c occur in the representation.

In retrospect, the existence of the construction (18) is not surprising, though the fact that it produces the missing series is. There is a development of the construction, however, which is not even with hindsight so obvious. It turns out that one can construct supersymmetric extensions of the Virasoro algebra in the space considered above, for some choices of the algebras g and h and representation λ. In particular if g is taken to be $g_1 \oplus g_1$ and h to be the diagonal g_1, and if λ is taken to be $\lambda_1 \oplus \psi$, where λ_1 is any representation of g_1 and ψ is its adjoint representation, then the construction above can be extended to one of the supersymmetric extensions:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{1}{12}\, cm(m^2-1)\,\delta_{m,-n} \tag{21a}$$

$$[L_m, G_r] = (\frac{m}{2} - r)\, G_{m+r} \tag{21b}$$

$$[G_r, G_s] = 2L_{r+s} + \frac{1}{3}\, c(r^2 - \frac{1}{4})\, \delta_{r,-s} \tag{21c}$$

where G_r is defined either for $r \in \mathbb{Z}$ (the Ramond algebra) or for $r \in \mathbb{Z} + \frac{1}{2}$ (the Neveu-Schwarz algebra). The G_r's are constructed as trilinears in the fermion fields.

These algebras too were analysed, in Friedan-Qiu-Shenker (1984), who showed that degenerate unitary representations can only occur for:

$$c = \frac{3}{2}\,(1 - \frac{8}{(n+2)(n+4)}) \tag{22}$$

with n integral, and a finite set of values of h associated to each c. Our construction produces unitary representations with all these values of c. These results are alluded to in Goddard-Kent-Olive (1985) and will be described more fully in a paper currently in preparation.

Our hope is that the mathematical entities I have described do in fact occur in physics. It would be gratifying were the Virasoro algebra representations, which we know arise in various spin models at the critical point, actually subrepresentations of a deeper structure, which might be extendable away from the critical point. We also see a role for these ideas in two-dimensional coset space theories, where there is a gauged subgroup H of the group G in which the fields take values.

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