Jerzy Lukierski

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TWO SUPERGEOMETRIES FROM FOURFOLD GRADINGS OF SUPERALGEBRAS

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1. INTRODUCTION

The canonical model of Riemannian geometry with constant curvature tensor components in the tangent space is provided by the symmetric Riemannian spaces, classified by Cartan [1]. The symmetric coset space $K = G/H$ is characterized locally by the following decomposition of the Lie algebra g of the group G (see e.g. [2]):

$$[h_a, h_b] = c_{ab}{}^{c}\, h_c$$

$$[h_a, k_\mu] = c_{a\mu}{}^{\nu}\, k_\nu \qquad a = 1,\dots,n \quad \mu = 1,\dots,d$$

$$[k_\mu, k_\nu] = c_{\mu\nu}{}^{a}\, h_a \tag{1}$$

where $\dim G = n+d$, $g = h \oplus k$. One can introduce dual Cartan one-forms (θ^a, θ^μ) and write the Cartan-Maurer equations for G as the Cartan structure equations for Riemannian space $K = G/H$. If we define the torsion and curvature two-forms

$$T^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = \frac{1}{2} T^\mu{}_{\rho\tau}\, \theta^\rho \wedge \theta^\tau$$

$$R^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\tau}\, \theta^\rho \wedge \theta^\tau \tag{2}$$

and choose $\omega^\mu{}_\nu = C_{a\nu}{}^{\mu}\, \omega_\mu$, one obtains using Jacobi identities that

$$T^\mu{}_{\rho\tau} = 0 \tag{3a}$$

$$R^\mu{}_{\nu\rho\tau} = - C_{a\nu}{}^{\mu}\, C_{\rho\tau}{}^{a} \tag{3b}$$

The vanishing torsion condition (3a) is implied by the Z_2 grading of the decomposition $g = h \oplus k$.

In order to obtain a general model of Riemannian geometry one should generalize the equation (3b) by assuming that the curvature tensor components $R^\mu{}_{\nu\rho\tau}$

depend on the points of the manifold, in a way consistent with Bianchi identities

$$dR^{\mu}{}_{\nu} + \omega^{\mu}{}_{\rho} \wedge R^{\rho}{}_{\nu} = 0 \tag{4}$$

The aim of this lecture is to show that one can introduce in an analogous way two supergeometries, by considering firstly two types of supersymmetric extensions of symmetric Riemannian spaces.

We show in Sec.2 that one can introduce two different types of supercosets corresponding, respectively, to Z_4 and $Z_2 \times Z_2$ gradings of superalgebra $\tilde{g}$. In Sec.3 we write corresponding decompositions of supersymmetric Cohen-Maurer equations. In Sec.4 we discuss two types of general supergeometries respectively Z_4 and $Z_2 \times Z_2$ gradings.

I would like to mention that this work has been done in collaboration with Hasiewicz. A more detailed discussion of the fourfold gradings of superalgebras is given in [3].

II. TWO TYPES OF FOURFOLD GRADINGS OF SUPERALGEBRAS

Let us consider simple superalgebra

$$\tilde{g} = g \oplus f \qquad \begin{array}{l} g \text{ - bosonic sector} \\ f \text{ - fermionic sector} \end{array} \tag{5}$$

Its bosonic sector is a sum $g = g_1 \oplus g_2$ of two commuting Lie algebras. For the physically interesting anti-de-Sitter superalgebras g_1 describes geometric and g_2 internal symmetry sector (see e.g. [4]):

$$\begin{array}{llll} d = 4: & OSp(N;4) & g_1 = O(3,2) & g_2 = O(N) \\ d = 5: & SU(2,2;N) & g_1 = O(4,2) & g_2 = U(N) \\ d = 7: & SU(4;N;H) & g_1 = O(6,2) & g_2 = Sp(N) \end{array} \tag{6}$$

Let us introduce, respectively, in g_1 and g_2 <u>two symmetric Riemannian pairs</u> (h_1,k_1), (h_2,k_2) where $g_i = h_i \oplus k_i$ $(i = 1,2)$ and

$$[h_i,h_i] \subset h_i \qquad [h_i,k_i] \subset k_i \qquad [k_i,k_i] \subset h_i \tag{7}$$

The relations (7) imply the existence of two involutions $\hat{I}_i$ $(\hat{I}_i^2 = 1)$, where

$$\hat{I}_1 h_1 = h_i \qquad \hat{I}_1 k_1 = -k_i \qquad i = 1,2 \tag{8}$$

Such involutions for the semisimple matrix Lie algebras have been classified by Cartan (see e.g. [5]).

Our first aim here is to generalize the notion of symmetric Riemannian pair for Lie algebra to the symmetric super-Riemannian quadruples for Lie superalgebras. We shall look for the splits of fermionic sector $f = (f_+, f_-)$ which implies that the superalgebra (5) divided as follows:

$$\tilde{g} = (h_1 \oplus h_2,\ k_1 \oplus k_2,\ f_+, f_-) \tag{9}$$

is endowed with fourfold graded structure. We can have the following two types of fourfold gradings:

a) Z_4 grading

$$\begin{array}{cccc} L_0 & L_1 & L_2 & L_3 \\ h_1 \oplus h_2 & f_+ & k_1 \oplus k_2 & f_- \end{array}\ , \tag{10}$$

where the graded Lie bracket $\langle \cdot,\cdot \rangle$ describing superalgebra S satisfies the relation

$$\langle L_i, L_j \rangle \subset L_{i+j} \qquad i,j = 0,1,2,3 \ (\mathrm{mod}\ 4)\ . \tag{11}$$

In Z_4 graded superalgebra one can introduce as the operator basis the generators from L_1. These generators close under fivefold graded bracket operation

$$[L_1, \{L_1, [L_1, \{L_1, L_1\}]\}] \subset L_1 \ . \tag{12}$$

b) $Z_2 \times Z_2$ grading

$$\begin{array}{cccc} L_{0,0} & L_{1,0} & L_{0,1} & L_{1,1} \\ h_1 \oplus h_2 & f_+ & f_- & k_1 \oplus k_2 \end{array}\ , \tag{13}$$

where

$$\langle L_{i,j}, L_{k,l} \rangle = L_{i+k,j+l} \quad (\mathrm{mod}\ 2)\ . \tag{14}$$

In the case of $Z_2 \times Z_2$ grading the superalgebra $\tilde{g}$ contains two supersymmetric subalgebras

$$\tilde{g}_\pm = (h_1 \oplus h_2,\ f_\pm) \tag{15}$$

which can be used for the construction of "conventional" non-linear realizations on the supersymmetric coset spaces $K_\pm = \tilde{G}/\tilde{G}_\pm$.

The notion of supersymmetric generalization of Riemannian symmetric pairs has been introduced in [6], and some examples were firstly given in [7]. Recently, the automorphisms of order four of classical Lie superalgebras were listed in [8].

3. MAIN THEOREM

We shall write down the general rules which provide for a given superalgebra $\tilde{g}$ and the choice of pairs of involvutions $(\hat{I}_1,\hat{I}_2)$ the answer whether

- fourfold grading does exist
- If it exists whether it is Z_4 or $Z_2 \times Z_2$ grading.

Because we shall be interested here in the fourfold gradings of the physical superalgebras (6) it is sufficient to consider only the matrix supercharges of the form

$$Q(\psi) = \left(\begin{array}{c|c} 0 & \psi^\sigma \\ \hline \psi & 0 \end{array}\right) , \tag{16}$$

where ψ denotes nxm-dimensional rectangular F-valued (F = R,C,H) matrix and $\psi \to \psi^\sigma$ is an involution with transposition. In such a case the bosonic charges are given by the set of matrices

$$\{Q(\psi),Q(\phi)\} \begin{pmatrix} A(\psi,\phi) & 0 \\ 0 & B(\psi,\phi) \end{pmatrix} \quad \begin{array}{l} A = \psi^\sigma\phi + \phi^\sigma\psi \\ B = \psi\phi^\sigma + \phi\psi^\sigma \end{array} . \tag{17}$$

We assume that the gradings under consideration are described by the substitution

$$\psi \to I_2\psi I_1 = \hat{I}\psi \qquad I_1^2 = \varepsilon_1 \qquad I_2^2 = \varepsilon_2 \qquad \hat{I}^2 = \varepsilon_1\varepsilon_2 = \varepsilon .$$

Further,

$$\psi^\sigma \to I_1^\sigma \psi^\sigma I_2^\sigma = (\hat{I}\psi)^\sigma \tag{18}$$

where we denote $I_1^\sigma I_1 = \eta_1$, $I_2^\sigma I_2 = \eta_2$. The involutions in the bosonic sector

$$\begin{aligned} \hat{I}_1 A &= I_1^{-1} A I_1 \\ \hat{I}_2 B &= I_2^{-1} B I_2 \end{aligned} \tag{19}$$

with the eigenstates

$$A_\pm = \frac{1}{2}(1 + \hat{I}_1)A, \qquad B_\pm = \frac{1}{2}(1 + \hat{I}_2)B \tag{20}$$

describe two symmetric Riemannian pairs (7).

One can introduce two eigenspaces $\psi_{\pm} = 1/2(1 \pm \hat{I})\psi$ of the involution $\hat{I}$ provided $\varepsilon = 1$. Further, because $\hat{I}Q(\psi) = Q(\hat{I}\psi)$ one can introduce two eigenstates $Q_{\pm}(\psi) = Q(\psi_{\pm})$ satisfying the conditions

$$\hat{I}Q_{\pm}(\psi) = \pm Q_{\pm}(\psi). \tag{21}$$

only provided $\varepsilon = 1$. One obtains

$$\{Q_{\pm}(\psi), Q_{\pm}(\phi)\} = \begin{pmatrix} A^{+}_{\eta}(\psi,\phi) & 0 \\ 0 & B^{+}_{\eta}(\psi,\phi) \end{pmatrix} , \tag{22}$$

where $\hat{I}_1 A^{+}_{\eta}(\psi,\phi) = \eta A^{+}_{\eta}(\psi,\phi)$, $\hat{I}_2 A^{-}_{\eta}(\psi,\phi) = \eta A^{-}_{\eta}(\psi,\phi)$. Similarly

$$\{Q_{\pm}(\psi) , Q_{\mp}(\phi)\} = \begin{pmatrix} A^{-}_{\eta}(\psi,\phi) & 0 \\ 0 & B^{-}_{\eta}(\psi,\phi) \end{pmatrix} , \tag{23}$$

where $\hat{I}_1 A^{-}_{\eta}(\psi,\phi) = -\eta A^{-}_{\eta}(\psi,\phi)$; $\hat{I}_2 A^{+}_{\eta}(\psi,\phi) = -\eta A^{+}_{\eta}(\psi,\phi)$. Identifying $Q_{\pm}(\psi)$ with $f_{\pm}$ (see (8)) we obtain for the superalgebras with the fundamental matrix representation of the odd sector given by Eq.(16) the following theorem:

i) the fourfold grading of $\tilde{g}$ does not exist if $\varepsilon = -1$ (arbitrary η), because $Q_{\pm}$ are not the eigenvectors of $\hat{I}$.

ii) The grading of $\tilde{g}$ is Z_4 if $\varepsilon = +1$, $\eta = -1$, because the relation (22) does not describe a sub-superalgebra.

iii) The grading of $\tilde{S}$ is $Z_2 \times Z_2$ if $\varepsilon = +1$, $\eta = +1$ because the relation (22) describes the sub-superalgebras $g_{\pm}$ (see (15)).

We would like to mention here that all the fourfold gradings for the superalgebras (6) has been given in [3].

4. TWO TYPES OF SUPERSYMMETRIC CARTAN STRUCTURE EQUATIONS DESCRIBING TWO TYPES OF SUPERCOSETS

Let us introduce the graded Cartan one-form $\tilde{\omega}$, dual to the vector fields satisfying the superalgebra $\tilde{g}$. Denoting by $\tilde{C}^{C}_{AB}$ the structure constants of $\tilde{g}$, one can write the supersymmetric Cartan-Maurer equation, describing the supergroup $\tilde{G}$

$$d\tilde{\omega}^{A} = \frac{1}{2} \tilde{C}^{A}_{BC} \cdot \tilde{\omega}^{B} \wedge \tilde{\omega}^{C} . \tag{24}$$

The equation (24) is the basic starting point of the supergroup manifold approach [9-11]. Using the notation from the formulae (10) and (13) one can decompose the one-form $\tilde{\omega} = \tilde{\omega}^{A} g_{A} = \tilde{G}^{-1} d\tilde{G}$ as follows:

$$\tilde{\omega} = \Omega + \psi_{+} + \psi_{-} + E , \tag{25}$$

where $\Omega \in h_1 \oplus h_2$, $E \in k_1 \oplus k_2$ and $\psi_{\pm} \in f_{\pm}$.

We define the covariant derivative with respect to $h_1 \oplus h_2$ as $D = d + ad\Omega$ ($ad\Omega$ means that the generators $h_1 \oplus h_2$ are taken in the adjoint representation). One can write the Cartan structure equations describing the product of two symmetric spaces with the holonomy algebra h_1 and h_2 as follows:

$$R = D\Omega = -\frac{1}{2} E \wedge E$$
$$T = DE = 0 \qquad (26)$$

These two spaces with constant curvatures, describing for the physical choices of the fourfold gradings the space-time and internal sector, are unified via supersymmetry in the following way:

a) Z_4 grading

$$R = D\Omega = -\frac{1}{2} E \wedge E - \psi_+ \wedge \psi_-$$
$$T = DE = -\frac{1}{2} (\psi_+ \wedge \psi_+ + \psi_- \wedge \psi_-) \qquad (27)$$

b) $Z_2 \times Z_2$ grading

$$R = D\Omega = -\frac{1}{2} E \wedge E - \frac{1}{2} (\psi_+ \wedge \psi_+ + \psi_- \wedge \psi_-)$$
$$T = DE = - \psi_+ \wedge \psi_- , \qquad (28)$$

where for both gradings the fermionic torsions have the form

$$T_{\pm} = D\psi_{\pm} = -\frac{1}{2} \psi_{\mp} \wedge E \qquad (29)$$

The equations (27-29) describe two ways of constructing the supersymmetric extensions of symmetric Riemannian spaces, represented by the coset $\tilde{K} = \tilde{G}/H_1H_2$. Every such supersymmetric extension is characterized by two bosonic curvatures and torsions ($\Omega = \Omega_1 \oplus \Omega_2$, $T = T_1 \oplus T_2$), and two fermionic torsions ($T = T_+ \oplus T_-$). The explicit formulae (27-29) determine the curvature and torsion constraints, induced by the respective fourfold gradings.

5. TWO TYPES OF GENERAL SUPERGEOMETRIES

a) Z_4 graded supergeometry

The covariant derivatives in (27-29) are defined with respect to $h = h_1 \oplus h_2$ subalgebra and we shall call them H-curvatures and H-torsions. One can also introduce full $\tilde{G}$ curvatures defined as follows [9-11]:

$$\tilde{R}^A = d\tilde{\omega}^A + \frac{1}{2} C^A{}_{BC}\ \tilde{\omega}^B \wedge \tilde{\omega}^C \tag{30}$$

satisfying for the supergroup manifold $\tilde{G}$ the condition $\tilde{R}^A = 0$, equivalent to (24). In the general case one can assume that $\tilde{R}^A \neq 0$, and the only restriction is provided by the Bianchi identities (see e.g. [11])

$$d\tilde{R}^A + C^A{}_{BC}\ \tilde{\omega}^B \wedge \tilde{R}^C = 0 \quad . \tag{31}$$

The $\tilde{G}$ curvatures are obtained by adding to H-curvatures and H-torsions suitable terms describing $\tilde{G}$ as super-Riemannian manifold. Our assumption describing the general Z_4 graded supergeometry can be expressed by the replacement of superalgebra structure constants $\tilde{C}^A{}_{BC}$ by the point-dependent functions. In such a way our supergeometry is described by a soft Lie superalgebra [12]. We assume that

α) The structure constants of the H-subgroup ($[L_0,L_0] \subset L_0$) and the H-covariance relations ($[L_0,L_i] \subset L_i$, $i = 1,2,3$) are not modified.

β) The remaining structure constants are becoming soft, i.e. deviate from the constant values given by the supergroup geometry.

It appears that the equations (27) and (29) retain their form provided that the right-hand sides are replaced by suitable arbitrary 2-forms. For example, writing

$$R = R^a h_a \ , \ E = E^\mu k_\mu \ , \qquad \psi_{\pm} = \psi^\alpha_{\pm} f_{\pm\alpha} \tag{32}$$

the first equation reads

$$R^a = -\frac{1}{2} R^a_{\mu\nu} E^\mu \wedge E^\nu + R^{a(+,-)}_{\alpha\beta}\ \psi_+{}^\alpha \wedge \psi_-{}^\beta \quad , \tag{33}$$

where $R^a_{\mu\nu}$ are arbitrary superfunctions on the superspace $\tilde{K} = \tilde{G}/H$ only restricted by Bianchi identities (31). We see there that

$$C_{\mu\nu}{}^a \to R_{\mu\nu}{}^a(z) \qquad C^{a(+,-)}_{\alpha\beta} \to R^{a(+,-)}_{\alpha\beta}(z) \quad , \tag{34a}$$

where $z \in \tilde{K}$. Similarly in the remaining equations we should make the replacement

$$C^{\mu(++)}_{\alpha\beta} \to R^{\mu(++)}_{\alpha\beta}(z), \ C^{(--)}_{\alpha\beta} \to R^{(--)}_{\alpha\beta}(z)$$

$$C^{\alpha(\mp)}_{\beta\mu} \longrightarrow R^{\alpha(\mp)}_{\beta\mu}(z) \ \ldots . \tag{34b}$$

It should be stressed that the replacement principles from ordinary to soft superalgebra structure which retains Z_4-grading introduce in a geometric way a large number of torsion and curvature constraints.

b) $Z_2 \times Z_2$ graded supergeometry

In an analogous way one can replace the equations (28) and (29) by the ones following from respective $Z_2 \times Z_2$ soft Lie superalgebra. We assume that

α) the structure constants of the H-subalgebra ($[L_{0,0}, L_{0,0}] \subset L_{0,0}$) and the H-covariance relations ($[L_{0,0}, L_{i,j}] \subset L_{i,j}$ where $i,j = 0,1$) are not modified.

β) The remaining structure constants are becoming soft, i.e. deviate from the constant values provided by the supergroup geometry.

In such a way again the equations (28) and (29) can be used as the structure equations of this type of supergeometry, with suitable substitutions of soft Lie superalgebra structure constants.

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INSTITUTE FOR THEORETICAL PHYSICS
UNIVERSITY OF WROCLAW
ul. CYBULSKIEGO 36
50-205 WROCLAW
POLAND