SOME APPLICATIONS OF TINY SEQUENCES

Andrzej Szymański

We introduce the notion of a tiny sequence (see the definition below) and we show that there exists a compact Hausdorff extremally disconnected separable space in which there exists a tiny sequence. The existence of such a space is used then to show that there is a compact Hausdorff separable space which is not co-absolute with dyadic space as well as to show that under Martin's axiom (MA) there is a separable closed subspace of ω^* ($=\beta\omega-\omega$) which is not a retract of $\beta\omega$. The first result gives a negative answer to a question raised by B.Efimov [2] and the second one generalizes a result obtained by M.Talagrand [8].

<u>1. Spaces with tiny sequences</u>. Let X be a topological space. We say that a sequence $\{P_n\}_{n\in\omega}$ of open families in X is a <u>tiny sequence</u> in X if

(i) $\bigcup P_n$ is dense in X for all $n\in\omega$, and

(ii) if F_n is a finite subfamily of $\mathbf{F}_n$ for each $n\in\omega$, then $\bigcup\{\bigcup F_n : n\in\omega\}$ is not dense in X.

Let us observe that if the cellularity of X, c(X), is uncountable, then there is a tiny sequence in X. Indeed, let P be an uncountable maximal family consisting of disjoint open subsets of X. Then $\bigcup P$ is dense in X and it suffices to set $P_n = P$ for all $n\in\omega$.

On the other hand, if either the π-weight of X, π(X), is countable or, under MA, if $c(X)\leq\omega$ and $\pi(X)<2^{\omega}$, then there are no tiny sequences in X. Indeed, let $\{P_n\}_{n\in\omega}$ be a sequence of open families in X such that $\bigcup P_n$ is dense in X for all $n\in\omega$. In the case $\pi(X)\leq\omega$ we proceed in the following way: let $\mathbb{P} = \{U_n : n\in\omega\}$ be a π-base in X. Let us choose $V_n\in P_n$ such that $V_n\cap U_n \neq \emptyset$, whenever $U_n \neq \emptyset$. Thus if we set $F_n = \{V_n\}$ for $n\in\omega$, $\bigcup\{\bigcup F_n : n\in\omega\}$ is dense in X.

In the case $c(X)\leq\omega$ and $\pi(X)<2^{\omega}$ and MA holds we proceed in the following way: by ccc of X we can choose a countable subfamily

Q_n from P_n such that $\bigcup Q_n$ is dense in X for each $n\in\omega$. Then consider the set S of all finite functions s such that $dom(s)\subset\omega$ and $s(n)\in Q_n$ for all $n\in dom(s)$. Partially order S by reverse inclusion. Then S is a ccc poset. Now, let B be a π-base in X such that $|B|<2^{\omega}$. For any $U\in B$, $U\neq\emptyset$, and $n\in\omega$ the sets $D_U=\{s\in S: \exists n\in dom(s)(s(n)\wedge U\neq\emptyset)\}$ and $D_n=\{s\in S: n\in dom(s)\}$ are dense subsets of S. Any generic subset of S yields a function t such that $dom(t)=\omega$, $t(n)\in Q_n$ for all $n\in\omega$ and $\forall\emptyset\neq U\in B\ \exists n\in\omega(t(n)\wedge U\neq\emptyset)$. Hence $\bigcup\{t(n): n\in\omega\}$ is a dense subset of X.

So, in both cases $\{P_n\}_{n\in\omega}$ cannot be a tiny sequence in X.

These two easy observations suggest the following question: does there exist a space with a tiny sequence which is ccc? or separable? We shall give an example of a separable space with a tiny sequence which is, in addition, compact Hausdorff and extremally disconnected. Before some next easy facts.

We say that a function $f: X\longrightarrow Y$ is <u>irreducible</u> if f is continuous, f(X) is dense in X and for every proper closed subset F of X, clf(F) is a proper subset of Y.

<u>Lemma 1</u>. Let $f: X\longrightarrow Y$ be irreducible. Then there is a tiny sequence in Y iff there is one in X.

Proof. Let $\{Q_n\}_{n\in\omega}$ be a tiny sequence in Y. Then the sequence $\{P_n\}_{n\in\omega}$, where $P_n=\{f^{-1}(U): U\in Q_n\}$ is a tiny sequence in X.

Let $\{P_n\}_{n\in\omega}$ be a tiny sequence in X. For $n\in\omega$, let $Q_n=\{U: U$ is open in Y and $f^{-1}(U)\subset V$ for some $V\in P_n\}$. Then $\{Q_n\}_{n\in\omega}$ is a tiny sequence in Y.

The embedding $i: X\longrightarrow Y$ is an irreducible map, whenever X is dense in Y. Hence

<u>Lemma 2</u>. If X is a dense subspace of a space Y, then there is a tiny sequence in Y iff there is one in X.

<u>Theorem 1</u>. There exists a separable compact Hausdorff extremally disconnected space in which there exists a tiny sequence.

Proof. Let $C[\omega^{\omega}]$ be the space of all compact non-empty subsets of the irrationals ω^{ω} equiped with the Pixley-Roy topology, i.e., by taking for each compact set $K\subset\omega^{\omega}$ all sets of the form $[K,U]=\{C: C$ is a compact subset of ω^{ω} and $K\subset C\subset U\}$, where U is open in ω^{ω}, to be a neighborhood base for K in $C[\omega^{\omega}]$. It is easily seen that the basic sets $[K,U]$ are closed-open and that $\bigcap\{[K,U]: U$ is an open neighborhood of $K\}=\{K\}$ for every K from $C[\omega^{\omega}]$. In consequence, the space $C[\omega^{\omega}]$ is T_1 and completely regular. We shall construct a tiny sequence in $C[\omega^{\omega}]$. Let us introduce a (standard) notation: ${}^{n}\omega=\{s: s: n\longrightarrow\omega\}$, $\omega^{<\omega}=$

$= \bigcup\{{}^n\omega: n\in\omega\}$ and if $s\in\omega^{<\omega}$, $D_s = \{t\in\omega^\omega: s\subset T\}$. Under this notation, for $n\in\omega$ let us define $P_n = \{[K, \bigcup\{D_s: s\in F\}]: K\in C[\omega^\omega]$ and F is a finite subset of ${}^{n+1}\omega\}$.

Claim. The sequence $\{P_n\}_{n\in\omega}$ is a tiny sequence in $C[\omega^\omega]$.

Since $\{D_s: s\in {}^n\omega\}$ is an open cover of the space ω^ω for every $n\in\omega$, each of the families P_n is even an open cover of $C[\omega^\omega]$. Now let us choose finite subfamilies F_n from P_n for each $n\in\omega$. Let $F_n = \{[K^n_1, \bigcup\{D_s: s\in F^n_1\}], [K^n_2, \bigcup\{D_s: s\in F^n_2\}], \dots, [K^n_{m_n}, \bigcup\{D_s: s\in F^n_{m_n}\}]\}$. For every $n\in\omega$ and $j\leqslant m_n$, F^n_j is a finite subset of ${}^{n+1}\omega$. Hence $E_n = \bigcup\{F^n_j: j\leqslant m_n\}$ is a finite subset of ${}^{n+1}\omega$ for each $n\in\omega$. There is a $t\in\omega^\omega$ such that $t|n+1\notin E_n$ for each $n\in\omega$. Indeed, since $|{}^1\omega| = \omega$ and $E_0\subset {}^1\omega$ is finite, there is a $t_0\in {}^1\omega - E_0$. Assume we have defined $t_n\in {}^{n+1}\omega - E_n$. Since $|\{s\in {}^{n+2}\omega: t_n\subset s\}| = \omega$, there is a $t_{n+1}\in {}^{n+2}\omega - E_{n+1}$ such that $t_n\subset t_{n+1}$. It suffices to put $t = \bigcup\{t_n: n\in\omega\}$. We shall show that $[\{t\},\omega^\omega]\cap\bigcup\{\bigcup F_n: n\in\omega\} = \emptyset$. Assume otherwise. Then there is a compact subset K of ω^ω and a $p\in\omega$ such that $t\in K$ and $K\in\bigcup F_p$. Hence there is a $j\leqslant m_p$ such that $K\in[K^p_j, \bigcup\{D_s: s\in F^p_j\}]$. This means that $K^p_j\subset K\subset\bigcup\{D_s: s\in F^p_j\}$. Since $t\in K$, there is an $s\in F^p_j$ such that $t\in D_s$. In consequence, $s\subset t$, so $t|p+1 = s\in F^p_j\subset E_p$. But $t|p+1\notin E_p$; a contradiction. Now, since $[\{t\},\omega^\omega]$ is a non-empty open subset of $C[\omega^\omega]$, $\bigcup\{\bigcup F_n: n\in\omega\}$ is not dense. The claim is proved.

The basic closed-open subsets of the space $C[\omega^\omega]$ can be divided into countably many centered families. To do this, let us take a countable open base B in the space ω^ω closed under finite unions. For $U\in B$ let $B_U = \{[K,V]: K\subset U\subset V\}$. Clearly, each of the families B_U is centered and each basic closed-open subset of $C[\omega^\omega]$ lies in some such B_U. In other words, we have shown that the space $C[\omega^\omega]$ has a σ-centered base. In consequence, there is a separable Hausdorff compactification of the space $C[\omega^\omega]$ (see [1]). Let $bC[\omega^\omega]$ be such a compactification. In virtue of Lemma 2 and of the claim, there is a tiny sequence in the space $bC[\omega^\omega]$. Taking the Gleason space over $bC[\omega^\omega]$ we get, by Lemma 1, the required space.

We have checked previously that spaces of countable π-weight are without tiny sequences. However we have constructed also a separable compact Hausdorff space with a tiny sequence. Hence

Corollary 1. There is a regular countable space with a tiny sequence.

Although there is a separable space with a tiny sequence, in general, for ccc spaces it is not the case.

Theorem 2. Dyadic spaces are without tiny sequences.

Proof. Let X be a dyadic space and let $\{P_n\}_{n\in\omega}$ be a sequence of open families in X such that $\cup P_n$ is dense in X for each $n\in\omega$. Since X is ccc, we can construct an open refinement Q_n of P_n such that $H_n = X - \cup Q_n$ is a closed nowhere dense and G_δ subset of X. Let $f: D^\kappa \longrightarrow X$ be a continuous surjection from the Cantor cube of weight κ. Then $f^{-1}(H_n)$ is a G_δ closed subset of D^κ for each $n\in\omega$. Hence there is a countable subset S of κ and there are closed sets $K_n \subset D^S$ such that $f^{-1}(H_n) = K_n \times D^{\kappa - S}$ for each $n\in\omega$. Let $A = \{a_n : n\in\omega\}$ be a countable dense subset of $D^S - \cup\{K_n : n\in\omega\}$. Then $A\times D^{\kappa-S}$ is a dense subset of $D^\kappa - \cup\{f^{-1}(H_n) : n\in\omega\}$. In consequence, $f(A\times D^{\kappa-S})$ is dense in $X - \cup\{H_n : n\in\omega\}$ and thus in X. For every $a\in A$, $f(\{a\}\times D^{\kappa-S})$ is a compact subset of X covered by each of the families Q_n. Hence, for each $n\in\omega$ there is a finite subfamily F_n of P_n such that $f(\{a_n\}\times D^{\kappa-S}) \subset \cup F_n$. Hence $f(A\times D^{\kappa-S}) \subset \cup\{\cup F_n : n\in\omega\}$ is dense in X. Therefore the sequence $\{P_n\}_{n\in\omega}$ cannot be a tiny sequence in X.

In virtue of Lemma 1, no irreducible preimage of a dyadic space cannot have a tiny sequence. In particular, the Gleason space over any dyadic space is without tiny sequence. In [7], B.Ponomariov and L.Šapiro posed the question (Problem 17), atributed to B.Efimov [2], as to whether every separable compact Hausdorff space is co-absolute with dyadic space. Since we have constructed a separable extremally disconnected compact Hausdorff space with a tiny sequence, this space cannot be co-absolute with dyadic space. So the Efimov-Ponomarion-Šapiro question has negative answer.

It is worthwile to mention that our compact Hausdorff separable space with a tiny sequence shows that there may be a space which is a continuous image of a space co-absolute with dyadic space but which is not co-absolute with dyadic space. Indeed, the convergent sequence is a dyadic space which is co-absolute with $\beta\omega$ and any separable compact space is a continuous image of $\beta\omega$.

It is tempting to show that compact Hausdorff spaces without tiny sequences are precisely those which are co-absolute with dyadic spaces. Although the answer to this conjecture is not known we shall indicate that to solve this positively we will need some additional axioms to ZFC. The following is for not hard to prove and is left to the reader as an exercise.

Proposition. Any compactification of the Suslin line has no tiny sequences and is not co-absolute with dyadic space.

2. Retracts of $\beta\omega$. It is a well known result, proved probably first by Gleason [3], that any separable compact Hausdorff extremally disconnected space can be embedded in ω^* as a retract of $\beta\omega$. The question arises, whether any separable closed subspace of ω^* is a retract of $\beta\omega$? It turns out that it depends on how these sets are placed in ω^*. M.Talagrand [8] was the first who gave an example of a separable closed subspace of ω^* which is not a retract of $\beta\omega$, but his construction heavily depends on the continuum hypothesis. In spite of Talagrand's example, any closed subspace of ω^* with countable π-weight is a retract of $\beta\omega$.

We are mainly interested on retracts of $\beta\omega$ which are P-sets in ω^*. This is motivated by the fact, proved in [5], that if for some regular uncountable cardinal κ, $U(\kappa)$ contains a P-set which is a retract of $\beta\kappa$, then there is a measurable cardinal in some transitive model of ZFC.

Let us start from recalling some basic facts concerning Boolean algebras and their Stone spaces. If B_1, B_2 are Boolean algebras and $h\colon B_1 \longrightarrow B_2$ is a homomorphism, there is a dual continuous map $h^*\colon St(B_2) \longrightarrow St(B_1)$ which is 1-1 iff h is onto and onto iff h is 1-1. Any right inverse to h, i.e., a homomorphism $g\colon B_2 \longrightarrow B_1$ such that $h\circ g = id_{B_2}$, is called a lifting to h. In this case $h^*\circ g^*$ is a retraction from $St(B_1)$ onto $h^*(St(B_2))$. We say that a Boolean algebra B has κ-separated property if whenever X and Y are subsets of B of cardinality less than κ such that $x \wedge y = 0$ for every $x \in X$ and $y \in Y$, then there is a $z \in B$ such that $x \wedge z = 0$ for every $x \in X$ and $y - z = 0$ for every $y \in Y$. We say that an ideal I of B is almost κ-complete if whenever $A \subset I$ is of cardinality less than κ, there is a $b \in I$ such that $a - b = 0$ for every $a \in A$.

Lemma 3. If I is an almost κ-complete ideal of a Boolean algebra B with a κ-separated property and the quotient algebra B/I is of cardinality $\leq \kappa$, then the quotient homomorphism $h\colon B \longrightarrow B/I$ has a lifting.

Proof. Let $\{c_\alpha : \alpha < \lambda\}$, $\lambda \leq \kappa$, enumerates B/I. We shall approximate the desired lifting by a sequence of homomorphisms on subalgebras of B/I. Thus define C_α, g_α, $\alpha < \lambda$, so that:

(1) each C_α is a subalgebra of B/I, $|C_\alpha| < \kappa$ and g_α is a homomorhism of C_α into B such that $h\circ g_\alpha = id_{C_\alpha}$,

(2) if $\alpha \leq \beta$, $C_\alpha \subset C_\beta$ and g_β extends g_α,

(3) if $\alpha < \lambda$, $c_\alpha \in C_\alpha$.

Conditions (1) - (3) guarantee that if we define $g = \bigcup\{g_\alpha:$

$:\alpha<\lambda\}$, g is a lifting to h.

Passing out to the construction, let us note that if we put $C_{\alpha+1}$ to be the subalgebra of $\mathbf{B}/I$ generated by $C_\alpha \cup \{c_{\alpha+1}\}$ and $C_\beta = \bigcup\{C_\alpha : \alpha<\beta\}$ for limit β, we will have $|C_\alpha|<\kappa$ and $c_\alpha \in C_\alpha$ for all $\alpha<\lambda$. The question remains only, how to extends g_α onto $C_{\alpha+1}$ to guarantee (1). To do this, assume $c_{\alpha+1} \notin C_\alpha$ and let $D = \{c \in C_\alpha : c \wedge c_{\alpha+1} = 0\}$ and $E = \{c \in C_\alpha : c - c_{\alpha+1} = 0\}$. Take $d \in B$ such that $h(d) = c_{\alpha+1}$. Then we put $X = \{d \wedge g_\alpha(c) : c \in D\}$ and $Y = \{g_\alpha(c) - d : c \in E\}$. Clearly, $X \cup Y \subset I$ and $|X \cup Y| < \kappa$. There is a $w \in I$ such that $v - w = 0$ for every $v \in X \cup Y$. There is also a $z \in B$ such that $z \wedge g_\alpha(c) = 0$ for every $c \in D$ and $g_\alpha(c) - z = 0$ for every $c \in E$. Hence for $a = (d-w) \vee (z \wedge w)$, $g_\alpha(c) \wedge a = 0$ for every $c \in D$ and $g_\alpha(c) - a = 0$ for every $c \in E$. It suffices to take $g_{\alpha+1}$ as the extension of g_α to $C_{\alpha+1}$ with the condition $g_{\alpha+1}(c_{\alpha+1}) = a$.

The space ω^* is the Stone space over the algebra $\mathcal{P}(\omega)/\mathrm{Fin}$. This algebra has ω_1-separated property. Under the continuum hypothesis for every almost ω_1-complete ideal $I \subset \mathcal{P}(\omega)/\mathrm{Fin}$, the space $\mathrm{St}((\mathcal{P}(\omega)/\mathrm{Fin})/I)$ is a retract of ω^* under the dual embedding. Every closed P-set in ω^* is the image of $\mathrm{St}((\mathcal{P}(\omega)/\mathrm{Fin})/I)$ under the dual embedding for some almost ω_1-complete ideal I in $\mathcal{P}(\omega)/\mathrm{Fin}$. Thus

Corollary 2 (J.van Mill [6]). (CH) Closed P-sets in ω^* are retracts of ω^*.

However not all (separable) closed P-sets in ω^* are retracts of $\beta\omega$. Tiny sequence is an obstacle.

Theorem 2. If X is a P-set in ω^* and there is a tiny sequence in X, then X cannot be a retract of $\beta\omega$.

Proof. We shall first prove the theorem in the case of X being dense in itself.

Suppose on the contrary that X is a retract of $\beta\omega$. Let $r : \beta\omega \longrightarrow X$ be a retraction and let $\{P_n\}_{n\in\omega}$ be a tiny sequence in X. Let $F_n = X - \bigcup P_n$. The sets F_n, $n \in \omega$, are closed and nowhere dense in X. For each $n \in \omega$, the set $E_n = \mathrm{cl}_{\beta\omega}(r^{-1}(F_n) \cap \omega)$ is a closed-open subset of $\beta\omega$ contained in $r^{-1}(F_n)$, and thus disjoint with X. Since X is a P-set in ω^*, there is a closed-open set $U \subset \beta\omega$ such that $X \subset U$ and $|E_n \cap U| < \omega$ for each $n \in \omega$. The set U is of the form $U = \mathrm{cl}_{\beta\omega} D$, where $D \subset \omega$. Hence $r(D)$ must be dense in X. Since $|E_n \cap D| < \omega$ and $E_n \cap D = r^{-1}(F_n) \cap E_n \cap D = r^{-1}(F_n) \cap D$, $|F_n \cap r(D)| < \omega$ for each $n \in \omega$. Now let $A = r(D) \cap \bigcup\{F_n : n \in \omega\}$ and $B = r(D) - \bigcup\{F_n : n \in \omega\}$. Let us decompose A into the following pieces: $A_0 = r(D) \cap F_0$ and $A_{n+1} = (F_{n+1} - F_n) \cap r(D)$ for $n \in \omega$.

Clearly, each of the sets A_n is finite and $A_n \subset \bigcup P_{n-1}$ for $n>0$. Hence it is possible to choose a finite subfamily S_n from P_n in such a way that $A_n \subset \bigcup S_{n-1}$ for every $n>0$, $n\in\omega$. Part B of the countable set r(D) is contained in the set $\bigcap\{\bigcup P_n : n\in\omega\}$. Hence for each $n\in\omega$ it is possible to choose precisely one member C_n from P_n in such a way that $B\subset C_0\cup C_1\cup\ldots$. Enlarging previously defined families S_n by adding C_n to S_n, we get a sequence $\{C_0\}$, $S_1\cup\{C_1\},\ldots,S_n\cup\{C_n\},\ldots$ of finite subfamilies of P_n's. Note that for every $d\in r(D)-A_0$ there is an $n\in\omega$ such that either $d\in C_0$ or $d\in C_n\cup\bigcup S_n$. Since r(D) is dense in X and A_0 is finite and X has no isolated points, $\bigcup\{\bigcup(S_n\cup\{C_n\}): n\in\omega\}$ is dense in X, which is a contradiction.

Passing out to the general case, assume again that X is a P--set in ω^* with a tiny sequence $\{P_n\}_{n\in\omega}$ and that X is a retract of $\beta\omega$. Hence X is separable and thus extremally disconnected and compact. Let $r: \beta\omega \longrightarrow X$ be a retraction and let I be the set of all isolated points of X. Then I is at most countable, so we can choose one member from each P_n in such a way that I will be contained in the union of all such chosen sets. Hence I cannot be dense in X. Let Z = X-clI. Then Z is a non-empty closed-open subset of X without isolated points. Clearly, Z is a P-set in ω^* and $r|r^{-1}(Z)$ is a retraction from $r^{-1}(Z)$ onto Z. Since $r^{-1}(Z)$ is a closed-open subset of $\beta\omega$, it is homeomorphic to $\beta\omega$ and Z lies as a P-set in the remainder of this copy of $\beta\omega$. It remains only to check that if we restrict the sequence $\{P_n\}_{n\in\omega}$ to Z we obtain a tiny sequence in Z. In virtue of the first part of the proof, this is a contradiction.

K.Kunen [4] showed that assuming MA, any extremally disconnected compact Hausdorff space of weight $\leq 2^{\omega}$ can be embedded in ω^* as a P-set. This together with Theorems 1 and 2 give

Corollary 3. (MA) There is a separable closed subspace of ω^* which is not a retract of $\beta\omega$.

The same conclusion under CH has been obtained by M.Talagrand [8]. Using, in addition, Corollary 2 we get

Corollary 4. (CH) There is a separable closed subspace of ω^* which is not a retract of $\beta\omega$ but which is a retract of ω^*.

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REFERENCES

1. van DOUWEN E. "Density of compactifications", preprint.
2. ЕФИМОВ Б.А. "О мощности расширений диадических пространств", Мат. сборник 96:4(1975), 615-632.
3. GLEASON A.M. "Projective topological spaces", Illinois Journal of Math. 2(1958), 385-398.
4. KUNEN K. "Some points in βN", Math. Proc. Cambridge Philos. Soc. 80(1976), 482-489.
5. KUNEN K., SZYMAŃSKI A., TALL F.D. "Baire irresolvable spaces and ideal theory", preprint.
6. van MILL J. "An introduction to $\beta\omega$", Vrije Universiteit Amsterdam, Rapport nr. 200.
7. ПОНОМАРЁВ В.И., ШАПИРО Л.Б. "Абсолюты топологических пространств и их непрерывных отображений", Успехи мат. наук. 31(1976), 121-136.
8. TALAGRAND M. "Non existence de relèvement pour certaines mesures finiement additives retractés de βN", Math. Ann. 256(1981), 63-66.

INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
BANKOWA 14
40-007 KATOWICE
POLAND