Pavel Šťovíček

Dirac monopole derived from representation theory

In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [301]--306.

Persistent URL: http://dml.cz/dmlcz/701320

DIRAC MONOPOLE DERIVED FROM REPRESENTATION THEORY

Pavel Šťovíček

1. INTRODUCTION

P.A.M. Dirac was the first who made the serious attempt to understand a point magnetic charge - Dirac monopole - in the framework of quantum mechanics [1]. He analyzed quantum mechanical motion of a charge particle in a magnetic monopole field and was led to the conclusion that a consistent quantum mechanical description exists only if the value of magnetic charge is quantized.

A new and very natural description of the monopole field is based on some concepts of modern differential geometry: the notions of vector potential and field strength are replaced by notions of connection in a complex line bundle and its curvature, respectively [7], [4].

We shall derive quantum kinematics of a charged particle in a magnetic monopole field assuming only the rotational symmetry of the configuration space. As the main tool for quantization on homogeneous spaces [2] we use Mackey's theory of systems of imprimitivity [5].

2. SYSTEMS OF IMPRIMITIVITY

The symmetry of the configuration space is expresed by a transformation group G of the space M, $G \times M \to M$. Its action $g \in G$, $g: M \to M : u \to gu$ should be represented in the quantum mechanical Hilbert space H by unitary operator $V: g \to V(g)$. Physical quantities represented by selfadjoint operators in H are transformed via $A \to V(g)AV(g)^{-1}$. The position measurements are described with the help of a projection-valued measure E on the Borel σ-algebra $\mathcal{B}(M)$ of M $E: S \to E(S)$ $S \in \mathcal{B}(M)$. The G-actions on M and the corresponding unitary transformations of H are bound together by

$$E(g.S) = V(g)\, E(S)\, V(g)^{-1} \quad \text{for all } g \in G,\ S \in \mathcal{B}(M).$$

A couple (E,V) fulfilling this condition is said to be an imprimitivity system. The notions of irreducibility and unitary equivalence are introduced in the same way as for unitary representations. If an element A of the Lie algebra is chosen, then due to Stone's Theorem exactly one selfadjoint operator P exists which satisfies

$$V(\exp t.A) = \exp(-itP/\hbar) \quad \text{for all real } t.$$

This operator is interpreted as a generalized momentum operator.

If the symmetry group G acts transitively on M Imprimitivity Theorem can be applied to classify all inequivalent irreducible imprimitivity systems. In this case the space M can be identified with the factor space G/K in a natural way, where K is the stability group of some fixed point in M.

Imprimitivity Theorem. Let G be a locally compact group satisfying the second countability condition, K its cloused subgroup. Let (E,V) be an imprimitivity system for G based on G/K. Then there exists a unitary representation L of the subgroup K such that (E,V) is equivalent to the canonical imprimitivity system (E^L, V^L) (definition see below). If L_1, L_2 are two unitary representations of K, then the corresponding canonical imprimitivity systems are equivalent if and only if these representations are equivalent. Finally, the canonical imprimitivity system is irreducible if and only if the representation L is irreducible.

Definition of the canonical imprimitivity system. Let us fix a quasiinvariant measure μ on G/K. The Hilbert space H consists of equivariant functions

$$\varphi : G \to H(L), \quad \varphi(ak) = L(k)^{-1}\varphi(a) \quad \text{for all } k \in K,$$

where H(L) is the Hilbert space of representation L, such that

1) mapping $a \to \langle \varphi(a), f \rangle$ is measurable on G for all $f \in H(L)$, where $\langle .,. \rangle$ is the Hermitian product in H(L),

2) $\|\varphi\| < \infty$, where the norm is induced by the Hermitian product

$$(\varphi, \varphi') = \int_{G/K} \langle \varphi(a), \varphi'(a) \rangle \, d\mu.$$

The operators E^L, V^L are given by :

$$E^L(S)\varphi = \chi_{\pi^{-1}(S)} \cdot \varphi ,$$

where χ_T is the characteristic function of a set T, $\pi : G \to G/K$ is the projection map,

$$[V^L(g)\varphi](a) = \sqrt{\frac{d\mu}{d\mu\circ g}}\,(g^{-1}.aK)\; \varphi(g^{-1}a) ,$$

where $d\mu / d\mu\circ g$ is Radon-Nikodym derivative and $\mu\circ g(S) := \mu(g.S)$.

3. CONSTRUCTION OF THE FIBRE BUNDLE

We consider the configuration space $\mathbb{R}^3 \setminus \{0\}$ and want to exploit its rotational symmetry. We shall consider only the angular part, the radial part will be inessential and so we shall investigate quantum kinematics on the unit sphere S^2 (embedded in $\mathbb{R}^3$). Points of S^2 can be identified with 2x2 traceless Hermitian matrices with determinant -1 :

$$u \in S^2 \leftrightarrow U = \sum u^k \sigma_k \quad ,$$

here σ_k , $k = 1,2,3$, are the Pauli spin matrices, $\sigma_k^2 = 1$, $\sigma_1\sigma_2 = i\,\sigma_3$ etc. The group SU(2) is the quantum mechanical symmetry group and it acts on S^2 according to

$$U \rightarrow T\,U\,T^* \quad , \; T \in SU(2).$$

To find the stability group K we fix the points $u_\pm = \pm\sigma_3 \equiv (0,0,\pm 1)$. Then $K = \left\{ \begin{pmatrix} \tau & 0 \\ 0 & \tau^* \end{pmatrix} \right\} = U(1)$, $|\tau| = 1$. Since SU(2) acts on S^2 transitively, S^2 can be identified with SU(2)/U(1) and Mackey's theory is applicable. So we have the Hopf fibre bundle $(SU(2), \pi, S^2; U(1))$. The projection map π is

$$\pi(T) = T\sigma_3T^* = (2\,\mathrm{Re}\,\alpha^*\beta, 2\,\mathrm{Im}\,\alpha^*\beta, \alpha\alpha^* - \beta^*\beta), \text{ if}$$

$$T = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \qquad , \; \alpha\alpha^* + \beta\beta^* = 1.$$

According to the Imprimitivity Theorem all inequivalent irreducible imprimitivity systems are in one to one relation with all inequivalent irreducible unitary representations of U(1) :

$$L_n : U(1) \rightarrow U(1) \;: \tau \rightarrow \tau^{-n}, \quad n \in \mathbb{Z}.$$

The Hilbert space H is constructed as the space of equivariant functions, i.e. complex functions on SU(2) satisfying the condition

$$\psi\left(T.\begin{pmatrix} \tau & 0 \\ 0 & \tau^* \end{pmatrix} \right) = \tau^n \psi(T)$$

and having the finite norm induced by the scalar product

$$(\psi_1, \psi_2) = \int_{S^2} \psi_1 \psi_2^* \, d\mu,$$

with invariant measure $d\mu = \sin\vartheta \, d\vartheta \, d\varphi$.

It is convenient to work in the associated complex line bundle instead of the principal bundle. The Hilbert space of equivariant functions is then replaced by unitarily equivalent Hilbert space of sections in the complex line bundle. In order to write down explicit relations, trivializations mappings of the Hopf bundle are necessary. Choosing an open covering of the sphere

$$\{U_+, U_-\}\ , \quad U_\pm = S^2 \setminus \{u_\pm\}\ ,$$

and using the spherical coordinates (ϑ, φ) we obtain two trivialization mappings determined by a pair of local sections :

$\rho_\pm : U_\pm \to \pi^{-1}(U_\pm) \subset SU(2),\ \pi \circ \rho_\pm = id_\pm$

$\rho_- : \vartheta, \varphi \to \alpha = e^{-i\varphi} \cos(\vartheta/2),\ \beta = \sin(\vartheta/2);\quad 0 < \vartheta \leq \pi,\ 0 \leq \varphi < 2\pi$

$\rho_+ : \vartheta, \varphi \to \alpha = \cos(\vartheta/2),\ \beta = e^{i\varphi} \sin(\vartheta/2);\quad 0 \leq \vartheta < \pi,\ 0 \leq \varphi < 2\pi,$

Hilbert space H' of sections consists of couples of functions (ψ_+, ψ_-), $\psi_\pm : U_\pm \to \mathbb{C}$, for which

$$\psi_-(u) = e^{-in\varphi} . \psi_+(u) \quad \text{almost everywhere on } U_+ \cap U_- .$$

To each equivariant function ψ there corresponds exactly one section $\psi_\pm = \psi \circ \rho_\pm$.

4.THE DIRAC MONOPOLE

The resulting (canonical) imprimitivity systems (E,V) based on S^2 involve the projection valued measure on Borel sets of the sphere

$$E(S) : (\psi_+, \psi_-) \to (\chi_{S_+}\psi_+, \chi_{S_-}\psi_-), \quad S_\pm = S \cap U_\pm\ ,$$

where χ_S is the characteristic function of $S \in \mathcal{B}(S^2)$. From V we shall determine generalized momentum operators. The one-parameter subgroup $\{\exp(-it\sigma_k/2)\}$ consists of all rotations around k-th axis. Therefore the operator J_k defined by

$$V(\exp(-it\sigma_k/2)) = \exp(-itJ_k/\hbar), \quad t \in \mathbb{R}\ ,$$

is interpreted as the k-th component of the angular momentum operator. For smooth equivariant functions ψ (i.e. $\psi_\pm$ are smooth), we have

$$(J_k\psi)(T) = i\hbar \frac{d}{dt} \psi(\exp(it\sigma_k/2).T) \Big|_{t=0}$$

and $\qquad J_k : (\psi_+, \psi_-) \to (J_{k(+)}\psi_+, J_{k(-)}\psi_-)$.

After straightforward calculation the explicit form of J_k is obtained:

$$J_{k(\pm)}\psi_\pm(u) = (-i\hbar X_k - q\,\alpha_\pm(X_k) - \tfrac{1}{2}n\hbar\, u^k)\,\psi_\pm(u).$$

Here $X_k = \varepsilon_{kjm}\, x^j \frac{\partial}{\partial x^m}$ is the vector field on the sphere induced by the assumed one-parameter group of actions;

$$\alpha_+ = \frac{n\hbar}{2q}(1 - \cos\vartheta)\, d\varphi\ , \quad \alpha_- = -\frac{n\hbar}{2q}(1 + \cos\vartheta)\, d\varphi$$

are 1-forms on U_+, U_- , respectively, satisfying (on $U_+ \cap U_-$) :

$$-i\frac{q}{\hbar}\alpha_- = -i\frac{q}{\hbar}\alpha_+ + (d\, e^{in\varphi})\, e^{-in\varphi}\ .$$

Hence the couple $(iq\alpha_+/\hbar, iq\alpha_-/\hbar)$ determines a connection in the complex line bundle. The 2-form

$$\beta = d\alpha_+ = d\alpha_- = n\frac{\hbar}{2q}\sin\vartheta\, d\vartheta \wedge d\varphi$$

is well defined on the entire sphere and can be interpreted as a magnetic field. Defining the magnetic charge g by

$$qg = \int_{S^2} q\beta = 2\pi n\hbar ,$$

we get

$$\vec{B}(\vec{u}) = \frac{g}{4\pi}\vec{u} ,$$

corresponding to a magnetic monopole placed in the centre of the sphere. At the same time we obtained Dirac's quantization condition

$$\frac{gq}{2\pi} = n\hbar , \quad n \in \mathbb{Z} .$$

If the vector potential $\vec{A}$ is used instead of the connection, the angular momentum operator can be written in the familiar form [6], [3] :

$$\vec{J} = \vec{r}\times(\vec{p} - q\vec{A}) - \frac{gq}{4\pi}\frac{\vec{r}}{r} .$$

5. CONCLUSION

The correct forms of quantum mechanical operators in the case of magnetic monopole are obtained from the assumption of rotational symmetry of the configuration space with the help of Mackey's theory.

ACKNOWLEDGEMENTS

I should like to thank Dr. J. Tolar for his help during the whole work.

REFERENCES

[1] DIRAC P.A.M. Proc. Roy. Soc. A, 133 (1931), 60-72

[2] DOEBNER H.D., TOLAR J. "Quantum mechanics on homogeneous spaces", J. Math. Phys., 16 (1975), 975-984

[3] FIERZ M. Helv. Phys. Acta, 17 (1944), 27

[4] GREUB W., PETRY H. R. "Minimal coupling and complex line bundles", J. Math. Phys., 16 (1975), 1347-1351

[5] MACKEY G. W. "Induced representations of groups and quantum mechanics", W. A. Benjamin, Inc., New York - Amsterdam (1968)

[6] POINCARÉ H. C. R. Acad. Sci., Paris, 123 (1896), 530

[7] WU T. T., YANG C. N. Phys. Rev. D., 12 (1975), 3845-3857

P. ŠŤOVÍČEK
DEPARTMENT OF MATHEMATICS
FACULTY OF NUCLEAR SCIENCE AND PHYSICAL ENGINEERING
CZECH TECHNICAL UNIVERSITY
HUSOVA 5
PRAGUE 1, CZECHOSLOVAKIA