András Recski

Applications of combinatorics to statics – a survey

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APPLICATIONS OF COMBINATORICS TO STATICS — A SURVEY

András RECSKI

Some more or less disconnected results on the applications of graphs and matroids to statics are summarized. The survey includes Maxwell's reciprocal figures for the Cremona diagrams, Maxwell's characterization of rigid planar frameworks in terms of the projection of polyhedra, the algorithmic characterization of generic rigidity in the plane by Laman, Lovász and Yemini, and the diagonal bracing of one-story buildings by Bolker and Crapo.

Although matroid theory is perhaps the most useful and promising tool in applications to qualitative problems in statics, here we use the terminology of graph theory only. (Of course, some of the proofs and/or algorithms, which are not presented here, depends heavily on matroids, but the survey can be read also by people with background in graph theory only.)

I Cremona-Maxwell diagrams

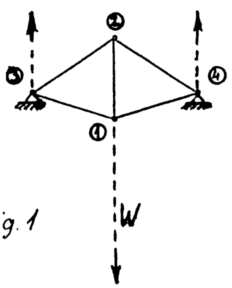
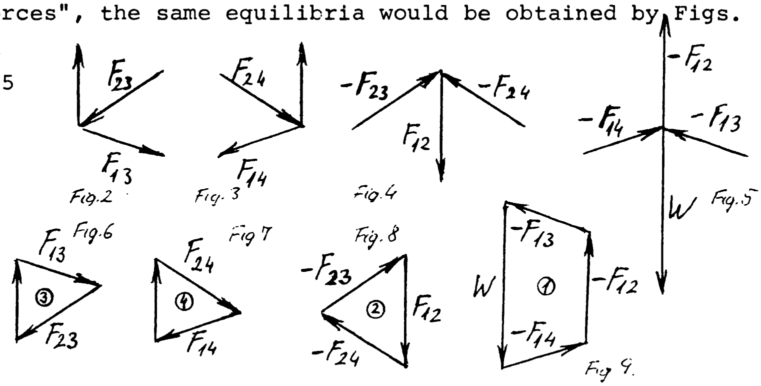


Fig. 1

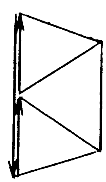
Consider the somewhat artificial "bridge" shown on Fig. 1. If it is loaded by the weight W at the point ①, two forces of -W/2 each must arise at points ③, ④, due to the symmetry. In order to have equilibrum at point ③ the vertical force -W/2 must be balanced by the forces F_{13} and F_{23} as shown on Fig. 2. Fig. 3 is almost the same with reference to point ④. Hence we see already that the rods 23 and 24 are under compression while the rods 13 and 14 are under tension. Accordingly, a tension F_{12} arises in the rod 12, as can be constructed on Fig. 4, for the equilibrum at point ② We have thus determined all the stresses in the rods and our calculation can be checked as on Fig. 5, by checking if the forces attacking point ① are in equilibrum.

Of course, if one prefers drawing "polygons of forces" instead of "stars of forces", the same equilibria would be obtained by Figs. 6-7-8-9 instead of Figs. 2-3-4-5 respectively. Observe, finally, that all the forces F_{12}, ..., F_{34} are drawn two times (once in each direction).



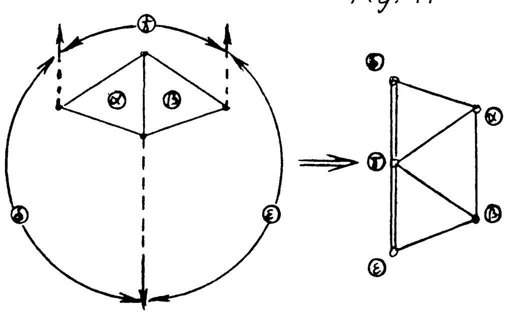
Hence, one can save "some work" by drawing all these "force-polygons" into a single, more complex picture, see Fig. 10. Here everything arises only once (and the directions of the forces, arising in the rods, disappear). This single drawing, containing all the information about the stresses, is called the *Cremona-diagram* (of the framework, with respect to this particular loading). Prior to the more recent analysis by computers, stresses in frameworks were usually calculated by this graphical method.

Fig. 10



Considering the framework as a graph G and its Cremona-diagram as another graph H one immediately sees that stars of G correspond to circuits of H. Hence it is not too surprising that a sort of "duality of planar graphs" can be introduced here. The usual method of constructing the dual of the drawing of a planar graph should be modified, however. Instead of having an "outer region" (to become a single vertex in the dual graph) here we need as many outer regions as the number of attacking forces. Fig. 11 explains the construction method, using what is usually called "Bow's notation". Observe, furthermore, the other difference: While in the "usual" constructions the corresponding pairs of edges are drawn perpendicular to each other, here they are parallel.

Fig. 11



From the point of view of the history of combinatorics one should realize that all these considerations are some 120 years old; they were several decades prior both to the mathematical development of dual graphs and to their applications in electric engineering.

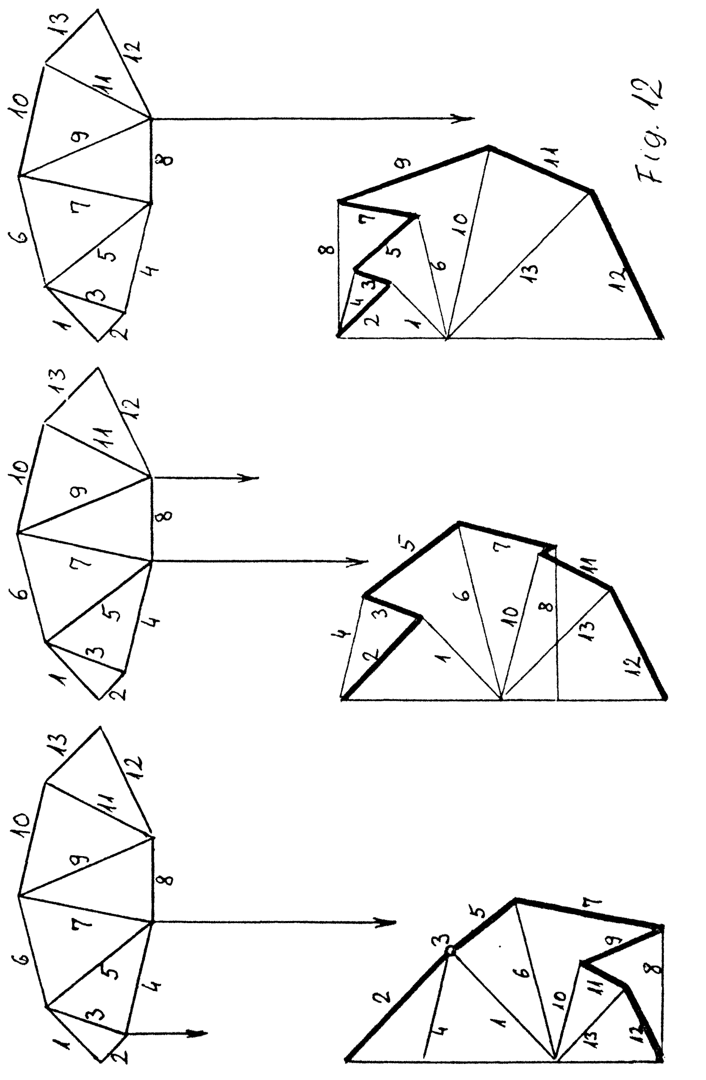


Fig. 12

We close this section by mentioning two further features of the Cremona-diagrams. First, it might happen that certain rods in a frameworks do not have any stress in case of a particular loading. Then two points coincide in the corresponding diagram. (E.g. rod 3 has no stress in the left-most example on Fig. 12.)

This somewhat more complicated example of Fig. 12 (where an asymmetric framework is loaded in three different ways, producing three different Cremona-diagrams) illustrates one more interesting phenomenon. The angles, formed by the lines of the Cremona-diagram inform us whether particular rods are under compression or under tension. For example, rod 7 is under tension and rod 9 is under compression in case of the first and second loadings while just vice versa in case of the third loading. This difference is reflected by the Cremona-diagrams where the corresponding edges form a V shape or a Λ shape respectively.

II Rigidity of planar frameworks and projections of polyhedra

The last remark in the previous section was already of qualitative character. In the rest of the paper all the results concentrate to a single qualitative question; whether a given framework is rigid or not. Even without giving a formal definition it is intuitively clear that the planar framework of Fig. 13 is rigid while that of

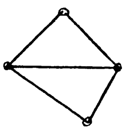


Fig.13

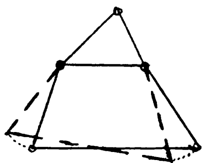


Fig. 14

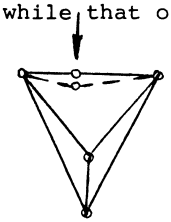


Fig.15

Fig. 14 is not. By definition, the framework of Fig. 15 will also be considered nonrigid, since the "infinitesimal motion" in the direction of the arrow is still possible. (This can be made precise by recollecting that a small deformation ε of a rigid body requires a force proportional to ε while in case of Fig. 15 the required force is proportional to ε^2 only.)

Keeping this in mind only the last framework of Fig. 16 is ri-

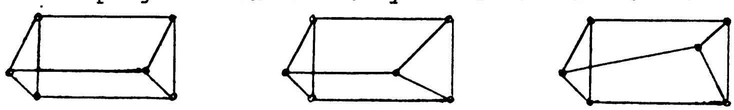
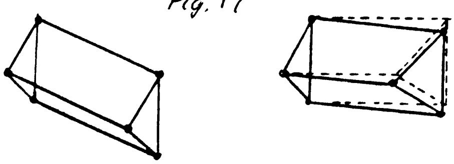


Fig. 16

Fig. 17



gid (though the second has infinitesimal motions only, see Fig. 17 as well). The difference between the last framework and any of the two former ones is that the first two arise as projections of 3-dimensional convex polyhedra while the third one does not. Similarly, only the last framework of Fig. 18 is rigid (the first two are projections again), also emphasizing that notions of *projective* geometry are very suitable for rigidity considerations.

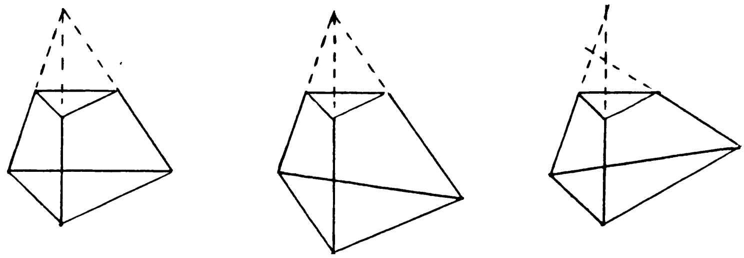


Fig. 18

These are examples of a general rule of Maxwell concerning such minimal (see below) planar frameworks which - as graphs - are isomorphic to the skeleton of convex polyhedra. Such a framework is rigid if and only if it does not arise as the projection of a 3-dimensional polyhedron. Minimality here means that the number j of joints and the number r of rods of the framework are related by $r=2j-3$. (All the planar frameworks with $r<2j-3$ are trivially nonrigid; rigid frameworks with $r>2j-3$ always have "unnecessary" rods as well. See also Section III below.

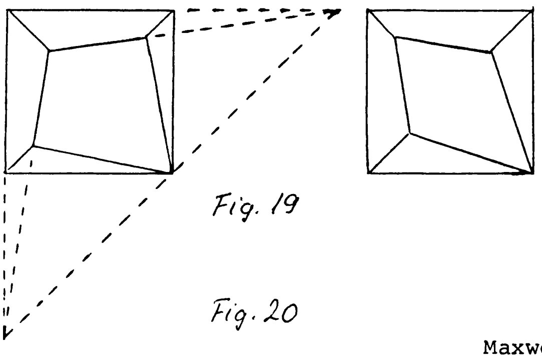
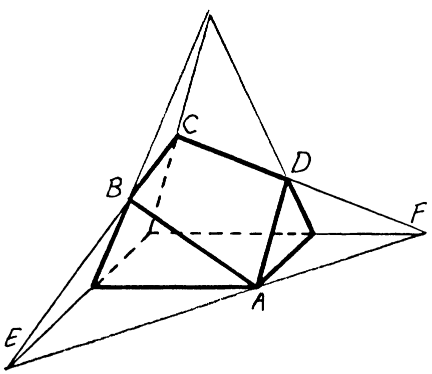


Fig. 19

Fig. 20



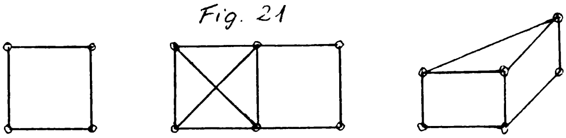
As an application of Maxwell's rule determine whether the planar frameworks of Fig. 19 are rigid. Of course, if they arise as projections of a polyhedron P then P should be a truncated pyramide, see Fig. 20. But A, B, C and D are coplanar if and only if A, E and F are collinear. Hence one can readily see that only the second of the two frameworks is rigid.

III Generic rigidity in the plane

In this section (apart from the last paragraph) all the frameworks are in the plane. Observe, however, that their graphs may be nonplanar.

Figs. 16, 18 and 19 presented 2 or 3 "similar" frameworks each (similarity means isomorphism in the sense of graph theory); some of them were rigid, some others not, depending on metric properties. Once again, it is intuitively clear (and can easily be made precise, see e.g. [Asimow-Roth, Crapo, Recski, Whiteley]) that if a pair F_1, F_2 of frameworks have isomorphic graphs and only F_1 is rigid then there must exist an algebraic relation among the length of the rods which is satisfied by F_2 but not by F_1. On the other hand, frameworks with graphs like on Fig. 21 are nonrigid, irrespective of the length of the individual rods. (In case of the first and the third graphs this is obvious since the relation $r \geq 2j-3$ of the previous section is violated.) Such graphs are called *generic nonrigid* while the graphs on Figs. 13, 15-19 are *generic rigid*, they can correspond to rigid frameworks if the lengths of the rods are suitably chosen. An alternative definition could be that a graph is generic rigid if, considering it as such a framework where the lengths of its edges are algebraically independent over the field of the reals, is rigid.

Fig. 21



Which graphs are generic rigid in the plane? The relation $r \geq 2j-3$ is obviously necessary but not sufficient (see the second graph on Fig. 21). Restricting ourselves to graphs with $r=2j-3$, situations like this can be excluded by requiring $r' \leq 2j'-3$ for every subgraph (with j' points and r' edges) of the graph as well. If this stronger condition is met, this implies already generic rigidity [Laman].

Since Laman's condition requires to check every subgraph, a direct algorithm to determine generic rigidity would be of exponential complexity as a function of the size $O(j+r)$ of the input.

Laman's condition was observed [Lovász and Yemini] to be equivalent to the following statement: A graph G with j vertices and $r=2j-3$ edges is generic rigid in the plane if and only if, doubling any edge e of G, the resulting graph G_e (with $2j-2$ edges) can be decomposed into the union of two trees. This is almost identical to a problem of [Nash-Williams]. Hence a polynomial algorithm (the matroid partition algorithm of [Edmonds]) can be applied.

Another result of Lovász and Yemini states that every 6-connected graph is generic rigid in the plane.

We close this section with two remarks on the 3-dimensional case. First, if a framework with j joints and r rods is minimally rigid then r=3j-6. The further condition r′≤3j′-6 for every subgraph with j′ points and r′ edges is also necessary but still not sufficient, see the counter-example [Asimow-Roth] on Fig. 22. A good characterization or possibly a polynomial algorithm for 3-dimensional generic rigidity seems to be the most interesting open problem in this field. Finally, Lovász conjectures that 10-connectedness implies 3-dimensional generic rigidity.

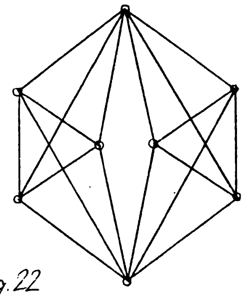
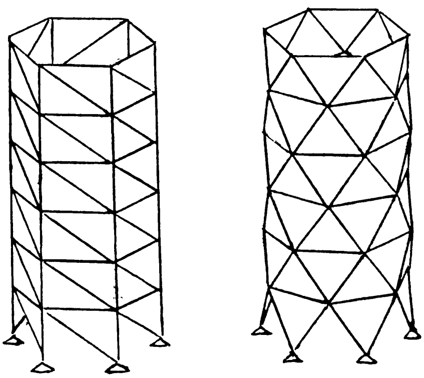


Fig.22

IV Diagonal bracing of one-story buildings

Fig. 23



The observations in this section are due to [Bolker and Crapo] but we present them without using matroid theory.

While - at least in the planar case - the genericity assumption helped to solve the problem of rigidity, one should also consider frameworks with rods of algebraically related lengths. (The examples in Section II were also of this character.) The application of pre-fabricated elements, panels etc. in today′s architecture even increases the need of such studies where certain distances are not algebraically independent, rather they are, say, equal. For example, both towers of Fig. 23 would contain a large number of identical building-blocks and only a careful analysis [Tarnai] shows that the second solution would be non-rigid.

Here we survey the rigidity results of some very simple structures, the so called one-story buildings. The reader will see, for example, that among the two "buildings" of Fig. 24 only one is rigid.

Consider the much simpler case of the square grids at first. Fig. 25 shows that such a planar framework can have a lot of defor-

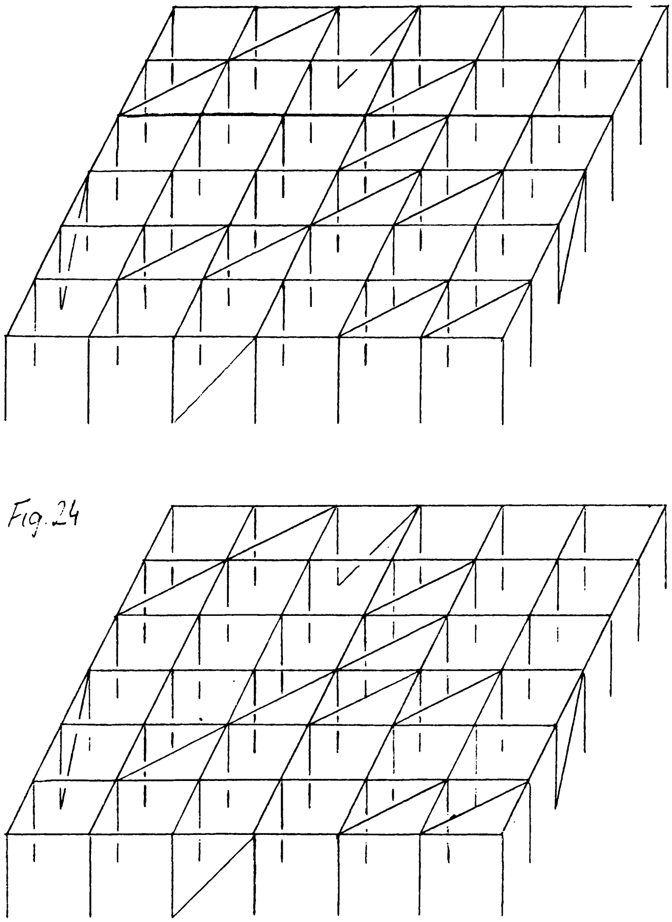


Fig.24

mations. If we applydiagonal braces on the squares, the deformations can be prevented. If certain squares have such diagonal braces, some others not, how can one decide whether the complete framework is rigid?

As it is intuitively shown on Fig. 25, deformations of the square grid can be obtained by combining elementary deformations of whole "rows" or "columns". If these rows and columns are denoted by letters and by numbers respectively (Fig. 26) then a diagonal brace, say at position a1, means that the actual deformation of row a and that of column 1 must be identical in size (hence the square at their intersection would only be rotated rather than deformed).

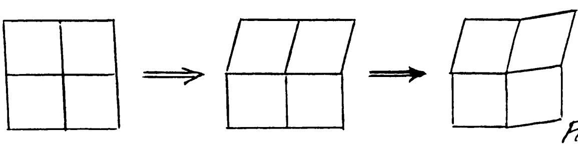
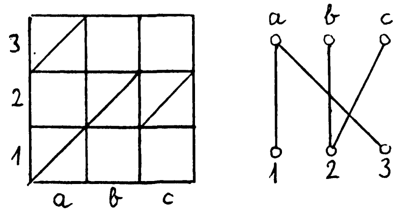


Fig.25

Thus, associating a bipartite graph (X,Y,E) with point set XUY and edge set E to the grid so that points of X and Y correspond to columns and rows, respectively, and edges to the diagonal braces (see Fig. 26), one can conclude that the grid is rigid

Fig.26



if and only if the bipartite graph is connected. In the example of Fig. 26 the graph is disconnected and the diagonal braces really cannot prevent a deformation like that of Fig. 27.

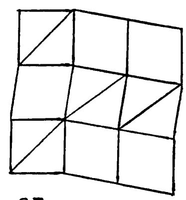


Fig. 27

While the connectivity of this associated bipartite graph completely characterizes the rigidity of the square grid, the solution of the real (3-dimensional) one-story building is more complicated. Some braces in vertical "walls" are obviously necessary (even bracing all the horizontal squares cannot prevent translations or rotations of the whole upper horizontal plane). Fig. 28 illustrates the effect of braces x,y; they prevent any motion of the planes S_x and S_y, respectively, along themselves. Hence simultaneously applying x and y the point P is completely fixed "to the space". If all the four outer vertical walls contain a diagonal brace each, as on Fig. 24, then the problem reduces to the bracing such a square grid where all of its corners are fixed "to the plane". (Fixing its corners prevents mechanical motions but infinitesimal deformations are still possible, see Figs. 30, 31 or 33.)

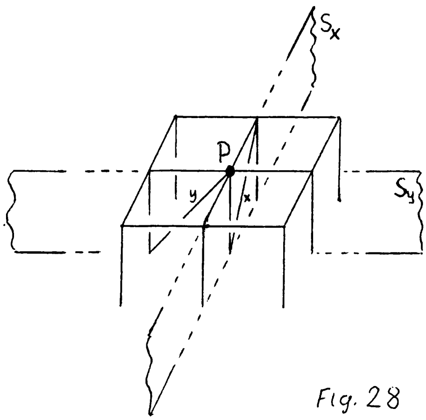


Fig. 28

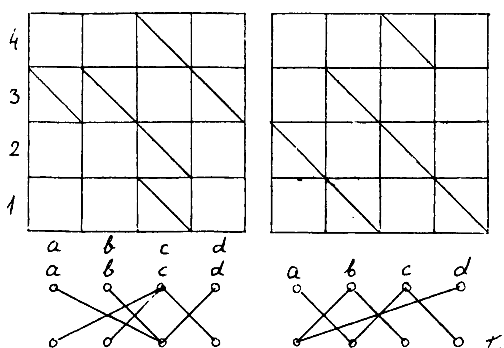


Fig. 29

What is the reason that the braces of the first example of Fig. 29 prevent the deformations while those of the second example do not? The associated bipartite graphs are 2-component forests in both cases, but the number of vertices in the bipartition classes (colour-classes) of the individual components are different in the first case and equal in the second. This turns out to be the crucial question. If the numbers and the letters (as vertices of the bipartite graph) can be paired so that the corresponding points are in the same component (like a2, b3, c4, d1 in the second example of Fig. 29) then the system of equations expressing that the deformation of, say column a is identical to that of row 2, that of column b to that of row 3 etc, implies the equation that the sum of the row-deformations is identical to that of the column-deformations. Both of these sums are zero (this is the mathematical meaning of fixing the corners) but in this case only one of these two extra constraints is really independent of the previous ones. Hence this system has a nontrivial solution in the second example of Fig. 29, see Fig. 30. Similarly, Fig. 31 shows that the first building of Fig. 24 is non-rigid.

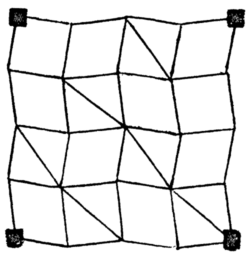


Fig. 30

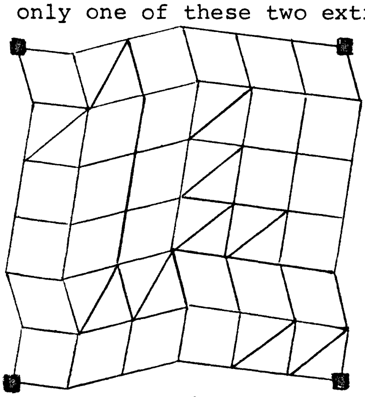


Fig. 31

These observations can very easily be generalized to one-story buildings of arbitrary rectangular (not necessarily square) shape. If four diagonal braces are applied on the "outer" vertical walls then the necessary and sufficient condition for the rigidity of a system of horizontal braces is that the associated bipartite graph contains such a 2-component forest where the ratios of the cardinalities of the colour-classes in the components are different from the ratio in the whole graph. E.g. this ratio is 2:3 in the whole graph in both examples of Fig. 32. In the first example the ratios are 2:3 for the components as well, while they are 1:1 and 1:2 in the two components of the second example. Accordingly, the first system has an infinitesimal deform.(Fig.33).

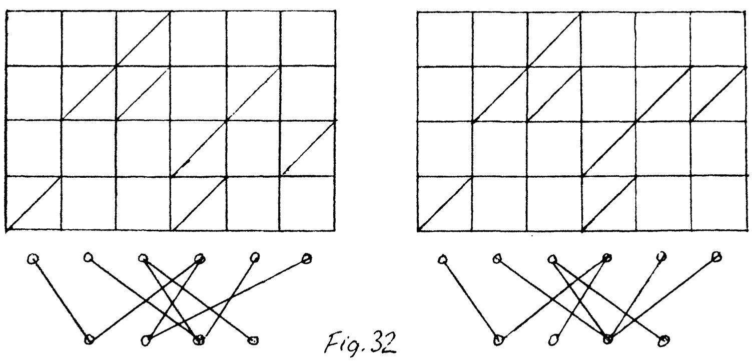
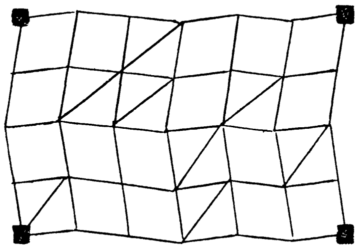


Fig. 32

Fig. 33



References

ASIMOW L. - ROTH B. "The rigidity of graphs", Part I. *Trans. Amer. Math. Soc.* 245 (1978) 279-289, Part II. *J. Math. Anal. Appl.* 68 (1979) 171--190.

BOLKER E. - CRAPO H. "How to brace a one-story building?" *Environment and Planning* B, 4. (1977) 125-152.

CRAPO H. "Structural rigidity", *Structural topology* 1 (1979) 26-45.

CREMONA L. *Le figure reciproche nella statica grafica*, Milano, 1879, in English: *Graphical Statics*, Oxford University Press, London, 1980.

EDMONDS J. Matroid partition, Math. of the Decision Sci. I, *Lectures in Applied Math.* II (1968) 335-345.

LAMAN G. "On graphs and rigidity of plane skeletal structures", *J.of Engineering Math.* 4 (1970) 331-340.

LOVÁSZ L. - YEMINI Y. "On generic rigidity in the plane", *SIAM J. Alg. Discrete Methods*, 3, 1 (1982) 91-98.

MAXWELL J. C. "On reciprocal figures and diagrams of forces", *Phil. Magasine*, 4, 27 (1864) 250-261.

NASH-WILLIAMS C. St. J. A. "Edge-disjoint spanning trees of finite graphs", *J. London Math. Soc.* 36 (1961) 445-450.

RECSKI A. "A network theory approach to the rigidity of skeletal structures, to appear in *Discrete Appl. Math.* (Parts I,II) *Structural Topology* (Part III).

TARNAI T. "Ciklikus szimmetriával rendelkező térbeli rácsos szerkezetek egyidejű statikai és kinematikai határozatlanságáról, *Építés- és Építészettudomány*, 11, 3-4 (1979) 241-257.

WHITELEY W. "Introduction to structural geometry" , *preprint*, 1977-8.

András RECSKI
Research Institute for Telecommunication
H-1026 Budapest
Gábor Á. u. 65.
H u n g a r y