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STRONGER ESTIMATES OF SMALLNESS OF SETS OF FRECHET NONDIFFERENTIABILITY OF CONVEX FUNCTIONS

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A well-known result of Asplund [1] says that every continuous convex function on a real Banach space X with a separable dual is Frechet differentiable at every point of some residual subset of X . This result was generalized by Robert [5] , who proved that every monotone operator on such a space X is single-valued and upper-semicontinuous at every point of some residual subset of X . The problem of finding more restrictive notions of smallness of the set of points of Frechet nondifferentiability of continuous convex functions on X was partially solved in [4] . There it was shown that this set is even σ-porous, but that it need not be a null set for any given Radon measure. Here we improve the results of [4] in two ways. Firstly, we introduce the notion of angle smallness of a subset of X , which is stronger than σ-porosity and we prove that for every monotone operator on a Banach space with a separable dual is single-valued and upper semicontinuous at all points of X except those which belong to some angle small set. Secondly, we introduce the notion of ball smallness and we show that for every ball small set in a Hilbert space there is a convex continuous function which is not Frechet differentiable at all points of this set. Although the notions of angle smallness and of ball smallness do not coincide, they seem to be rather close together and hence they give in Hilbert spaces rather good estimates of the magnitude of the sets of points of Frechet nondifferentiability of continuous convex functions.

Let X be a real Banach space. The open ball with the center $x \in X$ and the radius $r > 0$ is denoted by $B(x,r)$.

A set $M \subset X$ is said to be α-angle porous (where $\alpha > 0$) , if for every $x \in M$ and every $\varepsilon > 0$ one may find $z \in B(x,\varepsilon)$ and $f \in X^*$ such that

$$M \cap \{y \in X \ ; \ \langle y-z, f\rangle > \alpha \|f\| \, \|y-z\|\} = \emptyset .$$

If, for every α positive, M can be written as a countable union of α-angle porous sets, we say that M is angle small.

A set $M \subset X$ is called r-ball porous (where $r>0$) if for every $x \in M$ and every $\varepsilon > 0$ there is $u \in X$ such that $\|u-x\| = r$ and $M \cap B(u, r-\varepsilon) = \emptyset$. If M can be written as a union of a countable family of sets, each of which is r-ball porous for some $r>0$, we say that M is a ball small set.

It is easily seen that every ball small set in a separable Hilbert space is angle small and that, if a set M can be written as a union of a countable family of sets each of which is α-angle porous for every $\alpha > 0$, then M is a ball small set. This shows that, in separable Hilbert spaces, the notions of angle smallness and ball smallness are rather close.

Theorem 1. Let X be a real Banach space with a separable dual and let $T: X \to X^*$ be a monotone possibly multivalued operator with an arbitrary domain $D(T) = \{x;\ T(x) \neq \emptyset\}$. Then there is an angle small set $A \subset D(T)$ such that T is single-valued and upper-semicontinuous at every $x \in D(T) - A$.

Proof. We need to show that the set $A = \{x \in D(T) \ ; \ \lim_{\delta \to 0+} \operatorname{diam} T(B(x,\delta)) > 0\}$ is angle small. Let $A_n = \{x \in D(T) \ ; \ \lim_{\delta \to 0+} \operatorname{diam} T(B(x,\delta)) > n^{-1}\}$ and let C be a countable dense subset of X. Whenever $\alpha > 0$ is given, we decompose $A = \bigcup_{f \in C} A_{n,f}$ where $A_{n,f} = \{x \in A_n \ ; \ \operatorname{dist}(f, T(x)) < \alpha/2n\}$. Clearly, it suffices to prove that each of the sets $A_{n,f}$ is α-angle porous. Let $n, f, x \in A_{n,f}$ and $r > 0$ be fixed. There is $z \in X$, $\|z-x\| < r$ such that $\|T_z - f\| > 1/2n$ for some $T_z \in T(z)$. To show that $A_{n,f}$ is α-angle porous it is sufficient to prove that $A_{n,f} \cap \{y \in X \ ; \ \langle y-z, T_z - f\rangle > \alpha \|T_z - f\| \, \|y-z\|\} = \emptyset$. Whenever $y \in D(T)$ and $\langle y-z, T_z-f\rangle > \alpha \|T_z - f\| \, \|y-z\|$, and whenever $T_y \in T(y)$ we have $\langle y-z, T_y - f\rangle = \langle y-z, T_y - T_z\rangle + \langle y-z, T_z - f\rangle \geq \langle y-z, T_z-f\rangle >$ $> \alpha \|T_z - f\| \, \|y-z\| \geq \alpha \|y-z\| / 2n$. Hence $\|T_y - f\| \geq \alpha/2n$ for every $T_y \in T(y)$ which, according to the definition of the set $A_{n,f}$, shows that $y \notin A_{n,f}$.

Remark. Using the Gregorys method of separable reduction (cf. [2], p.141), one can recover the result of Kenderov [3], according to which every monotone non-empty valued operator $T: X \to X^*$ on an Asplund space X is single-valued and upper-semicontinuous at points of some residual subset of X.

Corollary 1. Let f be a continuous convex function on a real Banach space with a separable dual. Then the set of points at which f is not Frechet differentiable is angle small.

Theorem 2. Whenever M is a ball small set in a Hilbert space X, there is a continuous convex function f on X which is not Frechet differentiable at any point of M.

Proof. We assume first that M is a bounded r-ball porous set. Let $N = M \cup (X - B(0,R))$ where $R > 2r$ is chosen so that

(1) $\quad M \subset B(0,R-2r)$.

Let C be the closed convex hull of the set $\{(y,t) \in N\times R \ ; \ t \geq \|y\|^2\}$. Then the function $f(x) = \inf\{t \in R \ ; \ (x,t) \in C\}$ is easily seen to be well-defined, convex and continuous on X . It is also clear that

(2) $\quad f(x) \geq \|x\|^2$ for $x \in X$ and

(3) $\quad f(x) = \|x\|^2$ for $x \in M$.

We prove that f is not Frechet differentiable at any point of M. Whenever $x \in M$, we first note that the derivative of $\|y\|^2$ at x equals $2x$, hence (2) and (3) imply $2x \in \partial f(x)$. On the other hand, for every $\varepsilon > 0$ there is $u \in X$ such that $\|u-x\| = r$ and $B(u, r\sqrt{1-\varepsilon}) \cap M = \emptyset$. For $z \notin B(u, r\sqrt{1-\varepsilon})$ we have

$\|z\|^2 = \|x\|^2 + 2(x,z-x) - \|u-x\|^2 + \|z-u\|^2 + 2(u-x,z-x) \geq$
$\geq \|x\|^2 + 2(x,z-x) - \varepsilon r^2 + 2(u-x,z-x)$ which, together with $B(u, r\sqrt{1-\varepsilon}) \cap N = \emptyset$, shows that

$f(z) \geq \|x\|^2 + 2(x,t-x) - \varepsilon r^2 + 2(u-x,z-x)$ for every $z \in X$.

For $z = x + \varepsilon(u-x)$ we get

$f(z) - f(x) \geq 2(x,z-x) + \varepsilon r^2 = 2(x,z-x) + r^2\|z-x\|$,

which shows that f is not Frechet differentiable at x . To prove the general case we decompose $M = \bigcup M_n$, where each M_n is a bounded r_n -ball porous set and we construct the functions f_n by the above method. Since $0 \leq f_n(x) \leq A_n + \|x\|^2$ for some A_n , it suffices to put $f = \sum c_n f_n$, where c_n are sufficiently small positive numbers.

Remarks. 1. If a Lipschitz function $g: R \to R$ is not differentiable at every point of some residual subset of R then its graph serves as an example of a σ-porous set which is not angle small. Hence Corollary 1 gives a stronger result than Theorem 1 from [4] .

2. One can find a δ-convex surface in R^n which is not a ball small set. Hence (cf. [6]) one can find a convex function in R^n such that its points of nondifferentiability do not form a ball small set. Therefore ball smallness is not a characterization of the magnitude of sets of points of Frechet nondifferen-

tiability even in finite dimensional Hilbert spaces.

3. Any C^1-smooth surface in R^n which is not a countable union of δ-convex surfaces gives an example of an angle small set which is not contained in the set of points of nondifferentiability of any convex function on R^n. Hence angle smallness is also not a characterization of the magnitude of sets of points where a continuous convex function is not Frechet differentiable.

4. In one (very special!) case our notions give even an exact characterization of sets of points of nondifferentiability of convex functions. Namely, in a one-dimensional Banach space angle small as well as ball small sets are exactly countable ones.

There is quite a number of open problems in this area. Among them the following one seemed to us most intriguing. To motivate it, let us note that for any closed convex set C with empty interior the function $x \to \mathrm{dist}(x,C)$ is Frechet nondifferentiable at every point of C, and that the sets of points of Gateaux nondifferentiability of any continuous convex function on a separable Banach space can be covered by countably many δ-convex surfaces. (This is even a characterization of their magnitude, see [6]).

Problem 1. Can the set of points of Frechet nondifferentiability of a continuous convex function on a Banach space with a separable dual or on a separable Hilbert space be covered by countably many closed convex sets with empty interior and by countably many δ-convex surfaces?

Another very natural problem is, whether it is possible to recognize good differentiability properties of continuous convex functions on a Banach space using the given norm only.

Problem 2. Is there a notion of smallness of sets in separable Banach spaces with the following two properties?

a) If X is separable, then the norm on X is Frechet differentiable everywhere except points of some small subset of X.

b) If X is separable and its norm is Frechet differentiable everywhere except at points of some small subset of X, then X is separable.

In particular, is b) true with "small" meaning "angle small" or "σ-porous"?

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