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by

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1. Introduction. In [3] necessary and sufficient conditions in terms of Walsh - Paley martingales are obtained for existence of equivalent norms, uniformly convex (resp. uniformly differentiable) in every direction, in Banach spaces. In the same paper these results have been applied to obtain sufficient conditions for existence of such equivalent norms in Banach spaces X with total systems $F \subset X^*$ of arbitrary cardinality. In the present note we prove that the sufficient condition [3] for existence of equivalent norms, uniformly differentiable in every direction, is also necessary in WCG Banach spaces.

2. Definitions and results. The norm of a Banach space X is said to be uniformly differentiable in every direction if for any $x , y \in X$ with $\|y\| = 1$,

$$\lim_{t \to 0} t^{-1} \sup_{\|x\|=1} (\|x+ty\| + \|x-ty\| - 2) = 0.$$

A Banach space X is called weakly compactly generated (WCG) if X contains a weakly compact fundamental subset.

Let $\mathcal{G}$ be a family of subsets of a set M. We say that $\mathcal{G}$ is an uniformly finite covering of $A \subset M$ if there exists an integer k such that the union of any choice of different sets $\{G_i\}_{i=1}^{k} \subset \mathcal{G}$ contains A. We say that $\mathcal{G}$ is a σ- uniformly finite covering of A if $\mathcal{G}$ can be represented as a countable union of families, each one being an uniformly finite covering of A.

Proposition 2.1. Let X be a WCG Banach space whose norm is uniformly differentiable in every direction. Then, there exists a family $\mathcal{G}$ in X^* of symmetric convex weak* neighbourhoods of zero with the following properties :

(i) for each $x^* \in X^*$ there exists a $G \in \mathcal{G}$ and a number $a > 0$ so that $x^* \notin aG$,

(ii) $\mathcal{G}$ is a σ- uniformly finite covering of any bounded subset of X .

Lemma 2.2. Let X be a Banach space and $\{x_j\}_{j=1}^{i}$ X with

$$\max_{a_j = \pm 1} \Big\| \sum_{j=1}^{i} a_j x_j \Big\| < \varepsilon i .$$

Then, the conditions $x^* \in X^*$, $|x^*(x_j)| \geq 1$, $j=1,2,\dots,i$, imply $\|x^*\| \geq \varepsilon^{-1}$.

Lemma 2.3. Let $\{x_j\}_{j=1}^{i} \subset X$ be a basic sequence whose basis constant is equal to one such that $\|x_j\| = 1$, $j=1,2,\dots,i$ and $\sup\{t^{-1}(\|x+tx_j\| + \|x-tx_j\| -2);\ \|x\|=1,\ |t|\varepsilon i < 4, 1\leq j\leq i\} \leq \varepsilon/2$. Then, the equality $\varepsilon i < 4$ implies

$$\max_{a_j = \pm 1} \left\| \sum_{j=1}^{i} a_j x_j \right\| < \varepsilon i .$$

The proof of Lemma 2.3 is essentially that given in [2] .

Lemma 2.4. Let X be a WCG Banach space whose norm is uniformly differentiable in every direction. Then there exists a subset Z of the unit sphere of X, total over X^*, such that for any $\varepsilon>0$, Z can be represented as a countable union of sets $Z_i^{(\varepsilon)}$ so that the conditions $\{z_j\}_{j=1}^{i} \subset Z_i^{(\varepsilon)}$, $z_j \neq z_k$, $j \neq k$

$|x^*(z_j)| \geq 1$, $j=1,2,\dots,i$ for some $x^* \in X^*$, imply $\|x^*\| \geq \varepsilon^{-1}$.

Proof. We shall proceed by transfinite induction with respect to dens X .

If dens X $= \aleph_0$, then the assertion is trivial. Let dens X $= \aleph$ and suppose that Lemma 2.4 is true for each cardinal number less than $\aleph$. Since X is a WCG Banach space, then by a theorem of Amir and Lindenstrauss (cf. [1]) there exists a transfinite sequence of linear projections $P_\gamma : X \to X$, $0 \leq \gamma \leq \lambda$ so that $P_0 x = 0$, $P_\lambda x = x$ for all $x \in X$, $\|P_\gamma\| = 1$, $1 \leq \gamma \leq \lambda$, $P_\beta P_\gamma = P_\gamma P_\beta = P_{\min(\beta,\gamma)}$, $P_\gamma x \in \overline{(\bigcup_{\beta<\gamma} P_{\beta+1} x)}$ for all $x \in X$

and dens $P_\gamma X < \aleph$ for $0 \leq \gamma < \lambda$

Put

$$Y_\gamma = (P_{\gamma+1} - P_\gamma)X, \quad 0 \leq \gamma < \lambda .$$

Since Y_γ are WCG Banach spaces and dens $Y_\gamma < \aleph$, by the inductive hypothesis there exist sets $Z_\gamma \subset Y_\gamma$, $0 \leq \gamma < \lambda$ with the desired properties. Put

$$Z = \bigcup_{0\leq\gamma<\lambda} Z_\gamma .$$

It is easily seen that Z is total over X^*. Indeed, let $x^*(z)=0$ for all $z \in Z$. By transfinite induction we may prove that $x^*(P_\gamma x) = 0$ for each $x \in X$ and $\gamma \in [0,\lambda]$. Since $P_\lambda x = x$, then $x^*(x) = 0$ for each $x \in X$, i.e. $x^* = 0$.

Let $\varepsilon > 0$. Denote by S the unit sphere of X. Put

$S_i^{(\varepsilon)} = \{ x \in S;\ \sup t^{-1}(\|u+tx\| + \|u-tx\| - 2) < \varepsilon/2, u \in S, 0<t<4/\varepsilon i$
We shall prove that $S = \bigcup_{i=1}^{\infty} S_i^{(\varepsilon)}$. Suppose the contrary.

Then there exist $x \in S$, $u_i \in S$, $t_i \in (0, 4/\varepsilon i)$ so that

$$t_i^{-1}(\|u_i+t_i x\| + \|u_i - t_i x\| - 2) \geq \varepsilon/2 .$$

This, however, contradicts the fact that the norm of X is uniformly differentiable in every direction.

Let

$$Z_\gamma = \bigcup_k Z_{\gamma,k}^{(\varepsilon)} ,$$

where the conditions $y^* \in Y^*$, $|y^*(z_j)| \geq 1$, $j=1,2,\dots,k$, $\{z_j\}_{j=1}^{k} \subset Z_{\gamma,k}^{(\varepsilon)}$ imply $\|y^*\| \geq \varepsilon^{-1}$. Put

$$Z_{i,k}^{(\varepsilon)} = (\bigcup_\gamma Z_{\gamma,k}^{(\varepsilon)}) \cap S_i^{(\varepsilon)}.$$

Obviously,

$$\bigcup_{i,k} Z_{i,k}^{(\varepsilon)} = Z .$$

Let $x^* \in X^*$ satisfy $|x^*(z_j)| \geq 1$, $j=1,2,\dots,ik$, where $z_j \neq z_p$, $j \neq p$, $\{z_j\}_{j=1}^{ik} \subset Z_{i,k}^{(\varepsilon)}$. If we assume that there exist γ and $j_1, j_2, \dots, j_k$ such that $z_{j_1}, z_{j_2}, \dots, z_{j_k} \in Z_\gamma$,

then $\|y^*\| \geq \varepsilon^{-1}$, where y^* is the restriction of x^* to Y_γ. Thus,

$$\|x^*\| \geq \|y^*\| \geq \varepsilon^{-1} .$$

Otherwise, for each $\gamma < \lambda$ we have that

$$\operatorname{card}(\{j ;\ 1 \leq j \leq ik,\ z_j \in Z_{\gamma,k}^{(\varepsilon)}\}) < k .$$

Therefore, there exist $\gamma_1, \dots, \gamma_i$, $\gamma_p \neq \gamma_m$, $p \neq m$; $j_1, \dots, j_i$

with $z_{j_m} \in Y_{\gamma_m}$, $m=1,2,\dots,i$. Clearly, $\{z_{j_m}\}_{m=1}^{i}$ is a basic

sequence whose basis constant is equal to one. Hence, by the definition of $S_i^{(\varepsilon)}$ and Lemma 2.3, we obtain that

$$\max_{a_m = \pm 1} \| \sum_{m=1}^{i} a_m z_{j_m} \| < \varepsilon i .$$

In view of Lemma 2.2, this imply $\|x^*\| \geq \varepsilon^{-1}$, which concludes the proof.

2.5. Proof of Proposition 2.1. It suffices to denote by $\mathcal{G}$ the family, consisting of the sets

$$\{ x^* \in X^* ;\ |x^*(z)| \geq 1 \} ,$$

where $z \in Z$ and Z is the set, constructed in Lemma 2.4.

References

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