WHY EXTENDED NUMBERS IN SUPERSYMMETRY

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One of the main goals of particle physics today is the unification of the all interactions,including gravitational one.

It is believed that a neat way to solve these problems is to have an extended supersymmetric theory as the fundamental unifying theory (FERRARA S.).

The basic objects in supersymmetry (space-time symmetry which embeds bosons and fermions) are Grassmann variables θ_α , $\alpha = 1,\dots,4$. It is known that in the N extended supersymmetric theories these grassmanian variables have an internal symmetry index θ^i_α, i=1,...,N.

This is natural to ask on the physical motivation why extended numbers in supersymmetry and to study the connection between extended system of the anticommuting grassmanian numbers and extended supersymmetry.

The first physical motivation is that for supersymmetry some space-time dimensions D are special.Namely D=10 Yang-Mills theory (N=1 extended supersymmetry) is connected with D=4 Yang-Mills(N=4) and D=11 supergravity (N=1) is connected with D=4 (N=8) supergravity (SCHERK J.).

The reason for these special dimensions is in the algebraic properties of minimal spinors.

Supersymmetry in D dimensions is a theory invariant under simple supersymmetry algebra:

$$\{Q_{\hat\alpha}, Q_{\hat\beta}\} = 2(\Gamma^\mu)_{\hat\alpha\hat\beta} P_\mu ,$$

where the D matrices Γ^μ have dimensions $2^{[D/2]}$ and satisfy the Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} = 2\,\mathrm{diag}(\underbrace{+1,\dots,+1}_{t},\underbrace{-1,\dots,-1}_{s})$$

The Lorentz indices μ,ν run over all D while $\hat\alpha, \hat\beta$ are the indices of the Dirac matrices obeying the Clifford algebra.

It is shown (GLIOZZI F.et al.) that the possible properties of spinor $Q_{\hat\alpha}$: Majorana,Weyl,Dirac or Majorana-Weyl will depend on the dimension D: for and only for D=2,3,4 mod 8 there exists a Majorana representation of the Γ matrices. Then $Q_{\hat{\alpha}}$ can be defined as massive or massless Majorana spinors.

A Weyl spinor Λ is defined by

$$\Gamma^{D+1}\Lambda = \pm\Lambda ,$$

where the matrix Γ^{D+1} is given by

$$\Gamma^{D+1} = (-1)^{\frac{D-2}{4}} \Gamma^{0}\Gamma^{1}\ldots\Gamma^{D-1} .$$

We can have a Majorana-Weyl spinors if Γ^{D+1} is real, it is for D=2 mod 8 and so we obtain in particular special dimension D=10. For D=3 mod 8 we get special dimension D=11.

We can see from the following schema:

a) Eucledian space (t=0): $SO(1) \cong Sl(1;R)$,
$SO(2) \cong Sl_2(1;C)$,
$\overline{SO}(4) \cong Sl(1;H)^2$;

b) Minkowski space (t=1): $\overline{SO}(2,1) \cong Sl(2;R)$,
$\overline{SO}(3,1) \cong Sl(2;C)$,
$\overline{SO}(5,1) \cong Sl(2;H)$;

c) "conformal" space (t=2): $\overline{SO}(3,2) \cong Sp(4;R)$,
$\overline{SO}(4,2) \cong SU(2,2;C)$,
$\overline{SO}(6,2) \cong SO(4;H)$;

(R-real, C-complex, H-quaternions), that transverse dimension s-t is equal to the dimension of R,C,H.

From the Hurwitz theorem is known that the last extension gives octonions. We can see the following pattern:

s-t = 1 , 2 , 4 , 8;
R , C , H , O;
D = s+t = 3 , 4 , 6 ,10.

But the dimension D=10 is the last in the supersymmetric Yang-Mills theories (maximal spin 1).

The analogical connection we obtain in the supergravity theory, where the special dimension D=11 plays crucial role and after dimensional reduction to the D=4 this theory corresponds N=8 extended supersymmetry. N=8 is also the last step in the extended supergravity models, if we want to have maximal spin 2 for graviton.

In such a way we see that special dimensions D=10,11 in supersymmetry have deep connection with system of extended numbers.

The second physical motivation for using hypercomplex numbers in extended supersymmetries is the possibility for obtaining constrains, which reduce the number of ordinary fields in supermultiplets.

For example if we have a scalar superfield $\phi(x_\mu, \theta_\alpha^N)$, $\mu,\alpha=1,\ldots,4$ in N=8 extended supersymmetry we obtain $2^{4N} = 2^{32}$ ordinary fields. The problem is eliminating auxiliary fields in a realistic supersymmetric model.

One possibility for obtaining constraints is combination variables θ^i, i=1,...,N, into the complex (N=2) and the hypercomplex (N=4,8) grassmanian numbers. These constraints are "analytical", because they follow from the grassmanian analyticity (HRUBÝ J.):

$$\bar{D}\phi(x_\mu, \theta_\alpha^{N=1,2}) = 0 ,$$
$$\bar{\hat{D}}\phi(x_\mu, \theta_\alpha^{N=1,\ldots,4}) = 0 ,$$
$$\bar{\hat{\hat{D}}}\phi(x_\mu, \theta_\alpha^{N=1,\ldots,8}) = 0 ,$$

where $D = D^0 - iD^1$,

$\hat{D} = D^0 - e_a D^a$, e_a -quaternionic units,

$\hat{\hat{D}} = D^0 - o_i D^i$, e_i -octonionic units ,

$D^{i=0,\ldots,7} = \frac{\partial}{\partial\theta^i} + \frac{i}{2}\gamma.\partial\bar{\theta}^i$ are supercovariant derivatives.

The hypercomplex analyticity plays the following role of the constraint:

$$\hat{\phi}(x, \hat{\theta}, \bar{\hat{\theta}}) = \hat{\phi}(x + i\bar{\theta}^0\gamma\theta^a e_a - \frac{i}{2}\bar{\theta}^a\gamma\theta^b\varepsilon_{abc}e_c, \hat{\theta}) .$$

It means in the quaternionic case the following: the quaternionic supersymmetry can be realized in a smaller superspace, with hypercomplex Bose variable, but independently of $\bar{\hat{\theta}}$. This gives for the Taylor's expansion in $\hat{\theta}$ less indepedent coefficients.

The open problem is using octonions because they are nonassociative. There exists another way for N=8 extended supersymmetry: we can use complex quaternions combining variables θ^i, i=1,...,8 as follows

$$\eta^k = \theta^k + i\theta^{k+4} , \quad k = 1,\ldots,4 ;$$
$$\hat{\theta} = \eta^1 + e_a\eta^a , \quad a = 1,\ldots,3$$

Then we use the complex supersymmetric Cauchy-Riemann eqs.

$$D^1\phi^1(x,\eta,\bar{\eta}) = D^2\phi^2(x,\eta,\bar{\eta}) ,$$
$$D^1\phi^2(x,\eta,\bar{\eta}) = -D^2\phi^1(x,\eta,\bar{\eta}) ,$$

where $\phi(x,\eta,\bar{\eta}) = \phi^1(x,\theta,\bar{\theta}) + i\phi^2(x,\theta,\bar{\theta})$

and the supersymmetric quaternionic Cauchy-Riemann eqs.

$$D_0\phi_0(x,\hat{\theta},\hat{\bar{\theta}}) + D_a\phi_a(x,\hat{\theta},\hat{\bar{\theta}}) = 0\,,$$

$$D_0\phi_a(x,\hat{\theta},\hat{\bar{\theta}}) + D_a\phi_0(x,\hat{\theta},\hat{\bar{\theta}}) - \varepsilon_{abc}D_b\phi_c(x,\hat{\theta},\hat{\bar{\theta}}) = 0\,,$$

where $$\phi(x,\theta,\hat{\theta}) = \phi_0(x,\hat{\eta},\hat{\bar{\eta}}) + e_a\phi^a(x,\hat{\eta},\hat{\bar{\eta}})\,.$$

The combination of this analytical constraints gives many supersymmetric constraints on 2^{32} ordinary fields.

The last reason why to use extended numbers in supersymmetry is deep connection between hyper-Kähler geometry and supersymmetric KP(n) sigma models (LUKIERSKI J.,HRUBÝ J.).

On the end we want to say that the aim of this short communication was to give some attention on this interesting field in mathematical physics.

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