GROUPS OF DIFFEOMORPHISMS AND LIE THEORY

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0. There were many attempts to develope the "Lie theory" for groups of diffeomorphisms. In a paper of Leslie [6] the groups of diffeomorphisms are examples of so called Frechet Lie groups. Unfortunatelly, the category of the Frechet Lie groups seems to be too huge to treat because the implicit function theorem or the Frobenius theorem does not hold for Frechet manifolds in general case.
H. Omori in [7] (see also [2] and [8]) has introduced the category of so called ILH groups which seems to be better than the category of the Frechet Lie groups but the axioms are rather complicated and many fundamental problems remain unsolved.

Our aim in this note is to show the main resemblances and differences between the topological group structures of the Lie groups and the groups of diffeomorphisms.

1. It is well known that for a Lie group G the set $\mathfrak{g}$ of all one-parameter subgroups of G (i.e. continuous homomorphisms $X: \mathbb{R} \longrightarrow G$) has a Lie algebra structure such that for the exponential mapping $\exp: \mathfrak{g} \longrightarrow G$ ($\exp X = X(1)$) we have

$$\lim_{n\to\infty}(\exp(t/n)X \exp(t/n)Y)^n = \exp t(X+Y)$$

and

$$\lim_{n\to\infty}(\exp(-t/n)X \exp(-t/n)Y \exp(t/n)X \exp(t/n)Y)^{n^2} = \exp t^2[X,Y]$$

for all $X,Y \in \mathfrak{g}$ and $t \in \mathbb{R}$.

For a compact C^∞ manifold M (all manifolds in this note are supposed to be of class C^∞ and compact) the group D(M) of all C^∞ diffeomorphisms of M with the natural C^∞ Whitney topology is a topological group and each C^∞ vector field X on M generates an one-parameter subgroup $\mathbb{R} \ni t \longmapsto \operatorname{Exp} tX \in D(M)$ (a flow) of D(M) in the well-known manner.

The fact that all one-parameter subgroups of D(M) are of this form easilly follows from the following theorem of Bochner and Montgomery [1] :

<u>Theorem 1.</u> Let $\Phi: \mathbb{R}\times M \longrightarrow M$ be a continuous $\mathbb{R}$-transformation group on a manifold M of class C^k (analytic) and assume that for

each $a \in \mathbb{R}$ the mapping $M \ni x \longmapsto \Phi(a,x) \in M$ is of class C^k (analytic). Then Φ is of class C^k (analytic).

Hence for each one-parameter subgroup $X: \mathbb{R} \longrightarrow D(M)$ the mapping

$$M \ni x \longmapsto dX(t)(x)/dt\,|_{t=0} \in T_xM$$

is a well-defined C^∞ vector field which generates X.
Thus the set $\mathfrak{X}(M)$ of all C^∞ vector fields on M is the set of all one-parameter subgroups of D(M) .
It is well known that $\mathfrak{X}(M)$ has a natural Lie algebra structure and similar to the Lie groups case formulas are true:

Fact 1. For each $X, Y \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$ we have

$$\lim_{n\to\infty}(\mathrm{Exp}(t/n)X \cdot \mathrm{Exp}(t/n)Y)^n = \mathrm{Exp}\,t(X+Y)$$

and

$$\lim_{n\to\infty}(\mathrm{Exp}(-t/n)X \cdot \mathrm{Exp}(-t/n)Y \circ \mathrm{Exp}(t/n)X \circ \mathrm{Exp}(t/n)Y)^{n^2} = \mathrm{Exp}\,t^2[Y,X] .$$

Fact 1 immediately implies the following:

Fact 2. Let G be a closed subgroup of D(M) and let

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) : \mathrm{Exp}\,tX \in G \text{ for all } t \in \mathbb{R}\} .$$

Then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{X}(M)$.

In the Lie group case each closed subgroup is a Lie subgroup and similarly defined $\mathfrak{g}$ is its Lie algebra. In the case of D(M) we do not even know if the one-parameter subgroups from $\mathfrak{g}$ generate the connected component of the identity in G .

On the other hand each Lie subalgebra of the Lie algebra of a Lie group G generates a Lie subgroup of G. Is it true for Lie subalgebras of $\mathfrak{X}(M)$? More precisely, does the subgroup of D(M) generated by a Lie subalgebra L of $\mathfrak{X}(M)$ contain only one-parameter subgroups from L ?

The answer is "no" . For example, consider the Lie subalgebra L of $\mathfrak{X}(S^2)$ consisting of all vector fields which are tangent to the meridians on the upper half of the sphere. The subgroup $G \subset D(M)$ generated by Exp(L) contains one-parameter subgroups of D(M) which do not belong to L . In fact, we can go along the meridians from the equator to the lower half of the sphere, then rotate along a parallel and go up along the meridians, getting a rotation along the equator.

2. A difficult phenomenon in the case of the group $D(M)$ is that the mapping Exp is not even locally bijective :

Fact 3. The image $\mathrm{Exp}(\mathfrak{X}(M))$ of the mapping $\mathrm{Exp} : \mathfrak{X}(M) \longrightarrow D(M)$ does not contain any neighbourhood of the identity.

We shall prove the above fact only for the case $M=S^1$ (see [7]) whithout loss of generality, since we can extend diffeomorphisms from an imbedded in M circle onto the whole M.

Observe first that if a diffeomorphism $g \in D(S)$ with no fixed points is included in a flow then there is no pair $x,y \in S^1, x \neq y$ such that $\mathrm{dist}(g^n(x),g^n(y)) \longrightarrow 0$ as $n \longrightarrow \infty$, since each flow on S^1 with no fixed points is periodic.

Hence it suffices to construct a sequence (Ψ_n) of diffeomorphisms of S^1 such that

i) $\Psi_n \longrightarrow \mathrm{id}$ as $n \longrightarrow \infty$

ii) Ψ_n has no fixed points

iii) there are $x_n, y_n \in S^1$, $x_n \neq y_n$ such that $\mathrm{dist}(\Psi_n^k(x_n),\Psi_n^k(y_n)) \longrightarrow 0$ as $k \longrightarrow \infty$.

Put $\xi_n(e^{it}) = e^{i(t+2\pi/n)}$ and let (φ_n) be a sequence of diffeomorphisms of S^1 such that $\varphi_n \longrightarrow \mathrm{id}$ as $n \longrightarrow \infty$, $\mathrm{supp}\varphi_n \subset \{e^{it}: |t| < \pi/n\}$ and with $1 \in S^1$ as their attractor.

Then $\Psi_n = \xi_n \circ \varphi_n$ have the properties i)-iii) . It suffices to take x_n and y_n from a contracted by φ_n neighbourhood of $1 \in S^1$.

The mapping Exp is also not locally injective.

Consider the tore $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and a neighbourhood U of 0 in $\mathfrak{X}(T^2)$. There are natural n and m such that the vector fields $X(a,b) = (1/n, \sin 2\pi na/m)$ and $Y(a,b) = (1/n,0)$ lie in U . Since $\mathrm{Exp}\,tX(a,b) = (a+t/n, b+(\cos 2\pi a - \cos 2\pi(na+t))/2\pi m)$ and $\mathrm{Exp}\,tY(a,b) = (a+t/n,b)$, we have $\mathrm{Exp}X = \mathrm{Exp}Y$.

Fact 3 shows that it is reasonable to ask the following

Question. Does it exist a non-discrete subgroup G of D(M) such that the identity is the only one element of G which is included in a flow ?

3. Despite of the fact that Exp gives no chart on D(M) we can construct a mapping $\mathcal{E}xp : \mathfrak{X}(M) \supset V \longrightarrow D(M)$ which is a homeomorphism of a neighbourhood V of 0 in $\mathfrak{X}(M)$ onto a neighbourhood of the identity as follows.

Take a riemannian metric on M and denote by γ_v the maximal geodesic with the initial vector $v \in TM$. Let $U \subset TM$ be the family of all $v \in TM$ such that $\gamma_v(t)$ is defined for $|t| \leq 1$. U is an open subset of TM containing the 0-section. For $X \in \mathfrak{X}(M)$ with its image in U put $\mathcal{E}xp\, X(x) = \gamma_{X(x)}(1)$. One can prove the following:

<u>Fact 4.</u> There is a neighbourhood V of 0 in $\mathfrak{X}(M)$ such that

$$\mathcal{E}xp : V \longrightarrow D(M)$$

is a homeomorphism of V onto a neighbourhood of the identity in D(M).

Thus we have a chart in D(M) modelled on $\mathfrak{X}(M)$. Unfortunatelly the mapping $\mathcal{E}xp$ has not so nice algebraic properties as Exp. Nevertheless it allows us to prove the following fact about Exp:

<u>Fact 5.</u> $D_0(M)$ - the connected component of the identity in D(M) is topologically generated by $\mathrm{Exp}(\mathfrak{X}(M))$, i.e. the group generated by $\mathrm{Exp}(\mathfrak{X}(M))$ is dense in $D_0(M)$.

<u>Sketch of the proof.</u> It suffices to prove that every $g \in \mathcal{E}xp(V)$, where V is as in the thesis of Fact 4, is a limit of a sequence of compositions of elements from $\mathrm{Exp}(\mathfrak{X}(M))$.
Take $X \in V$ such that $\mathcal{E}xp\, X = g$. Put $X_\tau(x) = d\,\mathcal{E}xp\, tX(x)/dt_{|t=\tau}$ for $x \in X$ and $\tau \in [0,1]$. X is a C^∞ vector field on M and $X_0 = X$.
One can prove that

$$g = \mathcal{E}xp\, X = \lim_{n\to\infty} \mathrm{Exp}(1/n)X_{(n-1)/n} \circ \dots \circ \mathrm{Exp}(1/n)X_{1/n} \circ \mathrm{Exp}(1/n)X_0 .$$

Consider now the following problem:
Describe all continuous automorphism of $D_0(M)$.
Because of the Fact 5 we shall solve it, as in the classical Lie theory, by using a theorem concerning its Lie algebra (see [4], [9]).

<u>Theorem 2.</u> Each automorphism of the Lie algebra $\mathfrak{X}(M)$ is of the form g_* for a diffeomorphism $g \in D(M)$, where g_* is the natural action of the diffeomorphism g on vector fields.

Suppose now that A is a continuous automorphism of $D_o(M)$. It induces a bijection $\hat{A} : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ of the set of all one-parameter subgroups onto itself defined by the formula $\mathrm{Expt}\hat{A}(X) = A(\mathrm{Expt}X)$. By Fact 1 A is a Lie algebra automorphism. In fact,

$$\mathrm{Expt}\hat{A}(X+Y) = A(\mathrm{Expt}(X+Y)) = A(\lim_{n\to\infty}(\mathrm{Exp}(t/n)X\circ\mathrm{Exp}(t/n)Y)^n) =$$

$$= \lim_{n\to\infty} A((\mathrm{Exp}(t/n)X\circ\mathrm{Exp}(t/n)Y)^n) = \lim_{n\to\infty}(\mathrm{Exp}(t/n)\hat{A}(X)\circ\mathrm{Exp}(t/n)\hat{A}(Y))^n =$$

$$=\mathrm{Expt}(\hat{A}(X)+\hat{A}(Y)),$$

so $\hat{A}(X+Y) = \hat{A}(X)+\hat{A}(Y)$. Similarly $\hat{A}([X,Y]) = [A(X),A(Y)]$.
By Theorem 2 there is $g \in D(M)$ such that $\hat{A} = g_*$. Thus

$$A(\mathrm{Expt}X) = \mathrm{Expt}g_*(X) = g\circ\mathrm{Expt}X\circ g^{-1}.$$

Since $\mathrm{Exp}(\mathfrak{X}(M))$ topologically generates $D_o(M)$, $A(h) = ghg^{-1}$ for all $h \in D_o(M)$. We have proved :

<u>Theorem 3.</u> Each continuous automorphism of $D_o(M)$ has the form $h \longmapsto ghg^{-1}$ for a $g \in D(M)$.

It is interesting if all automorphisms of $D_o(M)$ are of this form.

<u>Question.</u> Do exist non-continuous automorphisms of $D_o(M)$?

4. There is a theorem of Epstein-Herman-Thurston (see [3], [5], [10]) which gives a stronger version of the Fact 5 :

<u>Theorem 4.</u> The group $D_o(M)$ is simple, i.e. it has no non-trivial normal subgroups.

<u>Corollary.</u> $D_o(M)$ is algebraically generated by $\mathrm{Exp}(\mathfrak{X}(M))$.

<u>Proof.</u> The group generated by $\mathrm{Exp}(\mathfrak{X}(M))$ is a normal subgroup of $D_o(M)$, since for $g \in D(M)$ and $X \in \mathfrak{X}(M)$ we have

$$g\circ\mathrm{Expt}X\circ g^{-1} = \mathrm{Expt}g_*(X).$$

One can ask how it is possible, unlikely to the Lie group case, that the Lie algebra $\mathfrak{X}(M)$ has many ideals (for example, the ideals L_x of all vector fields which are flat at a fixed point $x \in M$) and the group $D_o(M)$ is simple.
Despite of the fact that the group $D_x(M)$ generated by L_x is a proper subgroup of $D_o(M)$ (it contain only diffeomorphisms with x as a fixed point), $D_x(M)$ is not a normal subgroup.

Every Lie ideal of the Lie algebra of a Lie group G is an invariant subspace of the adjoint representation Ad of G ,since $Ad_{\exp X} = e^{ad_X}$. It is no longer true in infinite-dimensional case.For example,the Lie ideal L_x is not an invariant subspace of the adjoint representation of $D_0(M)$,sine $D_0(M)$ acts transitively on M .What is more (see [4], [9]) :

Theorem 5. Every non-trivial Lie ideal of $\mathcal{X}(M)$ is contained in L_x for some $x \in M$,so the Lie algebra $\mathcal{X}(M)$ has no non-trivial Lie ideals invariant under the adjoint action of $D_0(M)$.

The "right" Lie ideals of the Lie algebras of topological groups (i.e. ideals corresponding to normal subgroups) have to be invariant subspaces of the adjoint representations of the groups.
As we have seen in the example of a "bad" Lie subalgebra of vector fields on S^2,the "right" Lie subalgebras have to be invariant under adjoint action of their exponents.It is interesting if this condition is sufficient for ideals and subalgebras to be "right".

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