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HOMOTOPY AND HOMOLOGY IN PRETOPOLOGICAL SPACES

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With a process which is similar to the one of the classical case, we can develop a homotopy theory and a singular homology theory for pretopological spaces. This will be used in the continuation of this paper to obtain shape groups and **Čech homology** groups in another way. In other words, instead to approximate a topological space S by means of polyhedra, we reduce the more the set of admissible functions into S, in such a way to obtain the set of continuous maps.

In fact, among pretopological spaces, the ones given by means of principal filters (i.e. pf-spaces) are particularly interesting for us. The reason is that any topological space $(S,\mathcal{T})$ can be obtained as the inverse limit of a suitable directed family $\mathcal{S}=\{(S_i,\mathcal{P}_i)\}(i\in J)$ of pf-spaces such that $S_i=S$ for each $i\in J$. (We put $S_i>S_j$ iff the identity from S_i to S_j is precontinuous, etc.). Then, given a point $x\in S$, for any pf-space $S_i\in\mathcal{S}$, we can calculate the prehomotopy groups $\Pi_n(S,x)$, and we will put $\check{\Pi}_n(S,x) = \varprojlim \Pi_n(S_i,x)$. Similarly we obtain Čech homology groups.

We observe in advance that, if $(S,\mathcal{T})$ is a compact topological space, the shape groups $\check{\Pi}_n(S,x)$ and Čech homology groups are the classical ones. We only remark that it will be possible to consider finite open coverings and consequently to take pf-spaces belonging to the same homotopy type of finite graphs.

Here we expound a homotopy and homology theory for pretopological spaces, and we examine the properties of pf-spaces.

Finally we observe that finite graphs are pf-spaces and moreover a function from a topological space to a finite graph is precontinuous iff it is regular.

1. PRETOPOLOGICAL SPACES.

1.1 *Definition*(1) Given a nonempty set S, for each point $x\in S$ take a filter $\mathcal{F}_x$ in S such that $\dot{x}\leq \mathcal{F}_x$. The family $\mathcal{P}=\{\mathcal{F}_x\}(x\in S)$ is called a pretopology in S, and the couple $(S,\mathcal{P})$ is called a pretopological space. We also say "S carries

(1) See [6], page 1374. These spaces are called closure spaces in [1], pseudo-topological spaces in [5].

pretopology P", and we simply say "S is a pretopological space" when it is not necessary to specify P explicitly.
If each filter $F_x \in P$ is principal, we say that S is a principally filtered pretopological space or a pf-space.

Remark. Any topological space S may be considered as a pretopological space, putting for each $x \in S$ $F_x = U_x$, where U_x is the neighbourhood filter of x.

1.2 *Definition* Let (S,P) be a pretopological space. For any subset X of S, we put:

$$cl(X) = \{x \in S / F_x \wedge \dot{X} \neq 0\};$$
$$int(X) = \{x \in S / F_x \leq \dot{X}\}.$$

Remark. cl and int are respectively a closure operation and an interior operation such that $int(X) = S - cl(S-X)$ for any $X \subseteq S$. (S,P) is a topological space iff $cl(cl(X)) = cl(X)$ for any $X \subseteq S$.

1.3 *Definition* Let (S,P) and (S',P') be pretopological spaces, and $f:S \to S'$ a function. f is said precontinuous at $x \in S$ iff $f(F_x) \leq F'_{f(x)}$, where $f(F_x)$ denotes the filter in S' with base $\{f(F)\}(F \in F_x)$. Then we say that $f:S \to S'$ is a precontinuous map iff f is precontinuous at each $x \in S$.

Remark. Let us consider two subsets X of (S,P) and X' of (S',P'), and two points $x \in X$ and $x' \in X'$. By $f:(S,X) \to (S',X')$ we will denote a precontinuous map $f:S \to S'$ such that $f(X) \subseteq X'$. By $f:(S,X,x) \to (S',X',x')$ we will denote a precontinuous map $f:S \to S'$ such that $f(X) \subseteq X'$ and $f(x) = x'$.

1.4 *Proposition* Let (S,P) and (S',P') be pretopological spaces. $f:S \to S'$ is a precontinuous map iff:

$$(\forall A' \subseteq S')(\forall B' \subseteq S')(A' \cap cl(B') = \emptyset \Rightarrow f^{-1}(A') \cap cl(f^{-1}(B')) = \emptyset).$$

Proof: The necessity follows from $cl(f^{-1}(B')) \subseteq f^{-1}(cl(B'))$.
Sufficiency: Given $x \in S$ and $F' \in F'_{f(x)}$, we have $\{f(x)\} \cap cl(S'-F') = \emptyset$. Then $x \notin cl(f^{-1}(S'-F'))$, and so there is an $F \in F_x$ such that $F \cap f^{-1}(S'-F') = \emptyset$, i.e. $f(F) \subseteq F'$.

1.5 *Definition* Let S carry pretopology $P = \{F_x\}(x \in S)$, and $X \subseteq S$. Then let P^* be the family $\{F^*_x\}(x \in X)$, where $F^*_x = \{F \cap X\}(F \in F_x)$. We say that (X, P^*) is a subspace of the pretopological space (S,P).

1.6 *Definition* Let S' carry pretopology $P' = \{F'_x\}(x \in S')$, and S" carry pretopology $P'' = \{F''_y\}(y \in S'')$. We call product of (S',P') and (S'',P'') the couple (S,P), where $S = S' \times S''$, $P = \{F_{(x,y)}\}((x,y) \in S)$, and $F_{(x,y)}$ is the product filter of F'_x and F''_y.

2. HOMOTOPY OF PRECONTINUOUS MAPS.

As in the classical case, we can develop a homotopy theory.

2.1 *Definition* Let S and S' be pretopological spaces, and I the unit interval $\{t / 0 \leq t \leq 1\}$ with the standard topology. Two precontinuous maps f and g from S to S' are called homotopic (written $f \simeq g$) iff there exists a precontinuous map $H:S \times I \to S'$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in S$. Such a function H is called a prehomotopy of f to g, and we write $H:f\sim g$ to mean "H establishes a homotopy of f to g".

2.2 *Definition* Let S and S' be pretopological spaces. We say that S and S' belong to the same homotopy type, or more simply that S and S' are homotopic (written $S\sim S'$), iff we find two precontinuous maps $f:S\to S'$ and $g:S'\to S$ such that $gf\sim 1_S$ and $fg\sim 1_{S'}$.

2.3 *Definition* Let us consider a nonempty set S, a pretopological space (S',P') and a function $f:S\to S'$. For any $x\in S$, let us denote by F^*_x the filter of base $\{f^{-1}(F')\}(F'\in F'_{f(x)})$. The family $P^*=\{F^*_x\}(x\in S)$ is called the pretopology induced in S from (S',P') by f^{-1}.

2.4 *Proposition* If the function f from the set S to the pretopological space (S',P') is surjective, then (S,P^*) and (S',P') belong to the same homotopy type.

Proof: Clearly $f:(S,P^*)\to(S',P')$ is precontinuous. We obtain a precontinuous map $g:(S',P')\to(S,P^*)$, choosing for any $x'\in S'$ a point $g(x')\in f^{-1}(x')$. Clearly $fg=1_{S'}$. Moreover, since $F^*_x=F^*_{gf(x)}$ for any $x\in S$, we define a prehomotopy H of gf to 1_S putting:

$$H(x,t)=\begin{cases} x & \text{for } t=1,\ x\in S;\\ gf(x) & \text{for } t<1,\ x\in S.\end{cases}$$

3. PREHOMOTOPY GROUPS.

Let S be a pretopological space and $a\in S$. For each positive integer n, we consider the set $C_n(S,a)$ of all precontinuous n-loops based at a, i.e. the set of all precontinuous maps $f:I^n\to S$ such that $f(\dot{I}^n)=\{a\}$, where I^n is the unit n-cube with the standard topology and $\dot{I}^n$ is the boundary of I^n.

Then we define the homotopy of precontinuous n-loops based at a, and we give a group structure to the quotient set $\Pi_n(S,a)$. We call $\Pi_n(S,a)$ the n-dimensional prehomotopy group of S based at a.

The group $\Pi_n(S,a)$ is abelian for $n\geq 2$. Moreover, if b is another point of S and there exists a precontinuous path starting at a and ending at b, then the groups $\Pi_n(S,a)$ and $\Pi_n(S,b)$ are isomorphic.

Then, given $X\subseteq S$ and $a\in X$, we define the set $C_n(S,X,a)$ of relative precontinuous n-loops of the triple (S,X,a) and we construct the n-dimensional relative prehomotopy group $\Pi_n(S,X,a)$.

Afterwards we obtain a homomorphism $\partial:\Pi_n(S,X,a)\to\Pi_{n-1}(X,a)$.

Then, given two triples (S,X,a) and (T,Y,b) of pretopological spaces, we define an operation $*$, which associates to any precontinuous map $\phi:(S,X,a)\to(T,Y,b)$ a homomorphism $\phi_*:\Pi_n(S,X,a)\to\Pi_n(T,Y,b)$ for each dimension n.

So we obtain the prehomotopy system $\{\Pi,\partial,*\}$. The following six conditions are satisfied:

(I) If $\phi:(X,A,a)\to(X,A,a)$ is the identity, then $\phi_*:\Pi_n(X,A,a)\to\Pi_n(X,A,a)$ is the identical isomorphism.

(II) Let $\phi:(X,A,a)\to(Y,B,b)$ and $\psi:(Y,B,b)\to(Z,C,c)$ be precontinuous maps. Then $(\psi\phi)_*=\psi_*\phi_*$.

(III) Let $\phi:(X,A,a)\to(Y,B,b)$ be a precontinuous map and let $\psi:(A,a)\to(B,b)$ be the restriction of ϕ to A. Then the following diagram commutes:

$$\begin{array}{ccc} \Pi_n(X,A,a) & \xrightarrow{\partial} & \Pi_{n-1}(A,a) \\ \phi_* \downarrow & & \downarrow \psi_* \\ \Pi_n(Y,B,b) & \xrightarrow{\partial} & \Pi_{n-1}(B,b) \end{array}$$

(IV) Let us consider a triple (X,A,a) of pretopological spaces, the homomorphism $i_*:\Pi_n(A,a)\to\Pi_n(X,a)$ induced by the canonical injection $i:(A,a)\to(X,a)$, and the homomorphism $j_*:\Pi_n(X,a)\to\Pi_n(X,A,a)$ induced by the natural injection j from $C_n(X,a)$ to $C_n(X,A,a)$. The following sequence is exact:

$$\ldots\xrightarrow{\partial}\Pi_n(A,a)\xrightarrow{i_*}\Pi_n(X,a)\xrightarrow{j_*}\Pi_n(X,A,a)\xrightarrow{\partial}\Pi_{n-1}(A,a)\xrightarrow{i_*}\ldots\xrightarrow{\partial}\Pi_1(A,a)\xrightarrow{i_*}\Pi_1(X,a)$$

(V) If the precontinuous maps $\phi:(X,a)\to(Y,b)$ and $\psi:(X,a)\to(Y,b)$ are homotopic, then $\phi_*=\psi_*$.

(VI) If X={a}, all groups $\Pi_n(X)$ are null.

The proofs are similar to the ones given in the classical case (see for example [4]). Generally the fibration axiom is not valid; but this is not important for our next considerations.

4. HOMOLOGY GROUPS.

As in the classical case, we can develop a singular homology theory.

4.1 *Definition* Let us consider a pretopological space S and the standard euclidean p-simplex Δ_p. We call singular p-simplex (or more simply p-simplex) on S any precontinuous map from Δ_p to S.

We obtain the p-dimensional homology group $H_p(S)$ of the pretopological space S, and then, for any subset X of S, the p-dimensional relative homology group $H_p(S,X)$ of the pair (S,X). The boundary homomorphism $\partial:C_p(S)\to C_{p-1}(S)$ induces a homomorphism $\partial:H_p(S,X)\to H_{p-1}(X)$.

Afterwards, given two pretopological spaces S' and S", $X'\subseteq S'$ and $X''\subseteq S''$, we define an operation $*$, which associates to any precontinuous map $\phi:(S',X')\to(S'',X'')$ a homomorphism $\phi_*:H_p(S',X')\to H_p(S'',X'')$ for each dimension p.

So we obtain the homology system $\{H,\partial,*\}$. The following seven conditions hold:

(1) If $\phi:(X,A)\to(X,A)$ is the identity, then $\phi_*:H_p(X,A)\to H_p(X,A)$ is the identical isomorphism.

(2) Let $\phi:(X,A)\to(Y,B)$ and $\psi:(Y,B)\to(Z,C)$ be precontinuous maps. Then $(\psi\phi)_*=\psi_*\phi_*$.

(3) Let $\phi:(X,A)\to(Y,B)$ be a precontinuous map and $\psi:A\to B$ the restriction of ϕ to A. Then the following diagram commutes:

$$\begin{array}{ccc} H_p(X,A) & \xrightarrow{\phi_*} & H_p(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{p-1}(A) & \xrightarrow{\psi_*} & H_{p-1}(B) \end{array}$$

(4) Given a pair (X,A), let us consider the homomorphism $i_*:H_p(A)\to H_p(X)$ induced by the canonical injection $i:A\to X$, and the homomorphism $j_*:H_p(X)\to H_p(X,A)$ induced by the natural injection $j:Z_p(X)\to Z_p(X,A)$. The following sequence is exact:

$$\ldots\xrightarrow{\partial}H_p(A)\xrightarrow{i_*}H_p(X)\xrightarrow{j_*}H_p(X,A)\xrightarrow{\partial}H_{p-1}(A)\xrightarrow{i_*}\ldots\xrightarrow{\partial}H_o(A)\xrightarrow{i_*}H_o(X)\xrightarrow{j_*}H_o(X,A)$$

(5) If the precontinuous maps $\phi:(X,A)\to(Y,B)$ and $\psi:(X,A)\to(Y,B)$ are homotopic, then $\phi_*=\psi_*$.

(6) If $X=\{a\}$, then all homology groups $H_p(X)$ are null.

(7) Let A and U be nonempty subsets of a pretopological space X, such that $cl(U)\subseteq int(A)$. Then the canonical injection $i:(X-U,A-U)\to(X,A)$ induces an isomorphism $i_*:H_p(X-U,A-U)\to H_p(X,A)$.

The proofs of Conditions (1)-(6) are similar to the corresponding ones given for the classical singular homology (see for example [7]).

Proof of (7): For any p-simplex σ on the pretopological space X, $\{\widehat{\sigma^{-1}(\overset{\circ}{\widehat{X-U}})},\widehat{\sigma^{-1}(\overset{\circ}{\widehat{A}})}\}$ is an open covering of Δ_p, since $\{X-cl(U), int(A)\}$ is a covering of X, $X-cl(U)=$ $=int(X-U)$ and moreover $\sigma^{-1}(int(Y))\subset\widehat{\sigma^{-1}}(Y)$ for any $Y\subseteq X$.

Now let us denote by a_o, $a_1,\ldots,a_p$ the vertices of Δ_p and consider a baricentric subdivision $B^r(a_o,a_1,\ldots,a_p)=\Sigma\alpha_i\tau_i$ of the identity $(a_o,a_1,\ldots,a_p):\Delta_p\to\Delta_p$, with $r>0$ such that $\alpha_i\neq0$ implies $diam\tau_i<\varepsilon$, where ε is a positive real number such that either $V(y,\varepsilon)\subseteq\widehat{\sigma^{-1}}(\widehat{X-U})$ or $V(y,\varepsilon)\subseteq\widehat{\sigma^{-1}}(\widehat{A})$ for any $y\in\Delta_p$. Then, if $\alpha_i\neq0$, the simplex $\sigma_i=\sigma\tau_i$ on X is such that either $\sigma_i(\Delta_p)\subseteq X-U$ or $\sigma_i(\Delta_p)\subseteq A$.

Therefore, each element of $H_p(X,A)$ is represented by a relative p-cycle α which is a linear combination of p-simplices $\sigma\in C_p(X)$ such that either $\sigma(\Delta_p)\subseteq X-U$ or $\sigma(\Delta_p)\subseteq A$. So we can show that the homomorphism i_* is surjective and that $ker\, i_*=0$ (see [7], excision theorem).

Remark. Sometimes, for relative homotopy and relative homology, we will have to consider a subset X of a pretopological space (S,P) with a pretopology P_X finer then the one induced in X by P. We can repeat all foregoing considerations except excision theorem.

5. PF-SPACES.

Let S carry pretopology $P=\{\widehat{A_x}\}(x\in S)$, where $\widehat{A_x}$ denotes the principal filter of base A_x. From (S,P) we obtain a directed graph G whose vertex set is S. To define G we say that there is the directed arc xy iff $x\neq y$ and $y\in A_x$. (Instead, saying that there is the directed arc xy iff $x\neq y$ and $x\in A_y$) we obtain the dually directed graph G^*). G is called the graph of the pretopological space (S,P).

Viceversa, let G be a directed graph with vertex set S. For any vertex x, we consider the following sets:

$$A(x) = \{\{x\}\cup\{y\in S/\exists xy\};$$
$$A^*(x) = \{\{x\}\cup\{y\in S/\exists yx\}.$$

$P_G=\{\widehat{A(x)}\}(x\in S)$ and $P_G=\{\widehat{A^*(x)}\}(x\in S)$ are pretopologies on S.

Remark that, if G is the graph of a pf-space (S,P), then $\widehat{A(x)} = F_x$ for any

$x\in S$. So the pretopological spaces $(S,\mathcal{P}_G)$ and $(S,\mathcal{P})$ are isomorphic.

5.1 *Definition* A covering $\mathcal{R}$ of a pretopological space $(S,\mathcal{P})$ is an interior covering iff for each $x\in S$ there is an $X\in\mathcal{R}$ such that $X\in\mathcal{F}_x$ (see [1]).

5.2 *Definition* Let S be a pretopological space. S is compact iff any interior covering of S has a finite subcovering (see [1]). S is locally compact iff for each $x\in S$ we find a compact subspace X of S such that $X\in\mathcal{F}_x$.

5.3 *Definition* A function f from a pretopological space S to a directed graph G is o-regular (o*-regular) iff the following conditions hold:

(1) if v,w are distinct vertices of G and there is not the directed arc vw, then $f^{-1}(v)\cap cl(f^{-1}(w))=\emptyset$ (resp. $cl(f^{-1}(v)\cap f^{-1}(w)=\emptyset$);

(2) if X is a compact subset of S, then f(X) is finite.

5.4 *Proposition* Let us consider a pretopological space S, a directed graph G, and a function $f:S\to G$. We have the following three statements:

(a) Let the graph G be locally finite. If f is a precontinuous map from S to $(G,\mathcal{P}_G)$ (resp. to $(G,\mathcal{P}^*_G)$), then f is o-regular (resp. o*-regular).

(b) Let the pretopological space S be locally compact. If $f:S\to G$ is o-regular (resp. o*-regular), then f is a precontinuous map from S to $(G,\mathcal{P}_G)$ (resp. to $(G,\mathcal{P}^*_G)$).

(c) Let the graph G be finite. Then f is a precontinuous map from S to $(G,\mathcal{P}_G)$ (resp. to $(G,\mathcal{P}^*_G)$) iff f is o-regular (resp. o*-regular).

Proof: (a). Let G be locally finite and $f:S\to(G,\mathcal{P}_G)$ precontinuous.

(1). If $v\neq w$ and there is not the arc vw, we have $w\notin A(v)$. But $w\notin A(v) \Leftrightarrow v\notin cl(\{w\}) \Rightarrow f^{-1}(v)\cap cl(f^{-1}(w))=\emptyset$.

(2). For any vertex v of G, let us put $X_v=f^{-1}(A(v))$, and observe that $f^{-1}(v) \subseteq f^{-1}(int(A(v)) \subseteq int(X_v)$. Hence, for any $X\subseteq S$, the family $\{X_v\}(v\in f(X))$ is an interior covering of X. If X is compact, $\{X_v\}(v\in f(X))$ has a finite subcovering $\{X_{v_1},X_{v_2},\dots,X_{v_n}\}$; therefore $f(X) \subseteq \bigcup_{i=1}^{n} A(v_i)$. Hence f(X) is finite, because A(v) is finite for any $v\in G$.

(b). Let S be locally compact and $f:S\to G$ be o-regular.

For any subset Z' of $(G,\mathcal{P}_G)$, the family $\{f^{-1}(w)\}(w\in Z')$ is locally finite. In fact, for any $x\in S$ there is an $X\in\mathcal{F}_x$ which is compact, and $X\cap f^{-1}(w)\neq\emptyset$ only for a finite set of vertices $w\in G$, since f(X) is finite. Therefore (see [1]) $cl(f^{-1}(Z')) = \bigcup_{w\in Z'} cl(f^{-1}(w))$. Now, if $Y'\subset G$ is such that $Y'\cap cl(Z')=\emptyset$, we have $f^{-1}(v)\cap cl(f^{-1}(w))=\emptyset$ for any $v\in Y'$ and $w\in Z'$. So $f^{-1}(v)\cap cl(f^{-1}(Z'))=\emptyset$, and hence $f^{-1}(Y')\cap cl(f^{-1}(Z'))=\emptyset$.

(c). Let the graph G be finite. "Only if" part follows from (a). "If": observe that the family $\{f^{-1}(w)\}(w\in Z')$ is finite for any $Z'\subset G$.

Remark. Now suppose that the pf-space $(S,\mathcal{P})$ is such that, for any $x,y\in S$, $x\in A_y \Leftrightarrow y\in A_x$. In this case the pf-space is called simmetrical, and its graph is undirected.

Viceversa, let G be an undirected graph. We obtain a simmetrical pf-space $(G,\mathcal{P}_G)$

putting $P_G=\{\overline{vic\,x}\}(x\in G)$, where $vic\,x$ is the set containing x and all vertices y of G that are consecutive to x.

A function f from a pretopological space S to the graph G is called regular iff the following two conditions hold:

(1) $f^{-1}(v)\cap cl(f^{-1}(w)) = cl(f^{-1}(v))\cap f^{-1}(w) = \emptyset$ for any pair (v,w) of non consecutive distinct vertices of G;

(2) if $X\subset S$ is compact, then f(X) is finite.

We can prove that, if the undirected graph G is locally finite, the prehomotopy groups $\Pi_n((G,P_G),v)$ are the regular homotopy groups $Q_n(G,v)$ (see [3]).

6. EXAMPLES.

Given a covering $\mathcal{R}=\{A_i\}(i\in J)$ of a nonempty set S, we will define two pretopologies in S. To this purpose, we consider for each $x\in S$ the sets $A(x,\mathcal{R})=$ $=\cap\{A_i\in\mathcal{R}/x\in A_i\}$ and $St(x,\mathcal{R})=\cup\{A_i\in\mathcal{R}/x\in A_i\}$. Then we denote by $(S,R_\cap)$ and $(S,R_\cup)$ the pf-spaces, whose pretopologies are $R_\cap=\{\overline{A(x,\mathcal{R})}\}(x\in S)$ and $R_\cup=\{\overline{St(x,\mathcal{R})}\}(x\in S)$.

Now we define a function $\phi:S\to\mathcal{P}(J)$ putting $\phi(x)=\{i\in J/x\in A_i\}$ for any $x\in S$. Then we consider the set $Y=S_{/\rho}$, where ρ is the equivalence relation in S given by $x\rho x' \Leftrightarrow \phi(x)=\phi(x')$, and the canonical projection $\pi:S\to Y$. ϕ induces a function $\Phi:Y\to\mathcal{P}(J)$, which is given by $\Phi(y)=\phi(x)$ where $y\in Y$ and $x\in\pi^{-1}(y)$.

From the pf-space $(S,R_\cap)$ we obtain a directed graph $G^\cap$, whose vertex set is Y. To define $G^\cap$, we say that, given $y,y'\in Y$, there is the directed arc yy' iff $\Phi(y)$ is a proper subset of $\Phi(y')$.

Instead, from the simmetrical pf-space $(S,R_\cup)$ we obtain an undirected graph $G^\cup$, whose vertex set is Y. To define $G^\cup$, we say that $y,y'\in Y$ are consecutive iff $y\neq y'$ and $\Phi(y)\cap\Phi(y')\neq\emptyset$.

6.1 *Proposition* The pretopological spaces $(S,R_\cap)$ and $(G^\cap,P_{G^\cap})$ are homotopic.
Proof: In fact $R_\cap$ is the pretopology induced in S from $(G^\cap,P_{G^\cap})$ by π^{-1}.

6.2 *Proposition* The pretopological spaces $(S,R_\cup)$ and $(G^\cup,P_{G^\cup})$ are homotopic.
Proof: In fact $R_\cup$ is the pretopology induced in S from $(G^\cup,P_{G^\cup})$ by π^{-1}.

Now assume that the covering $\mathcal{R}=\{A_i\}(i\in J)$ is star-finite, and for any $j\in J$ denote by $\psi(j)$ the finite set $\{i\in J/A_i\cap A_j\neq\emptyset\}$.

6.3 *Proposition* The graph $G^\cap$ is locally finite.
Proof: Let $y,y'\in Y$.
In $G^\cap$ there is the arc y'y iff $\Phi(y')\subset\Phi(y)$. But $\Phi(y)$ is finite; so the set $\{y'\in Y/\exists y'y\}$ is finite.
Then in $G^\cap$ there is the arc yy' iff $\Phi(y)\subset\Phi(y')$. But $\Phi(y')\subseteq\psi(j)$ for any $j\in\Phi(y)$; so also the set $\{y'\in Y/\exists yy'\}$ is finite.

6.4 *Proposition* The graph $G^\cup$ is locally finite.
Proof: Let y and y' be distinct vertices of $G^\cup$. $y'\in vic\,y$ iff $\Phi(y)\cap\Phi(y')\neq\emptyset$; therefore $\Phi(y')\subseteq\psi(j)$ for some $j\in\Phi(y)\cap\Phi(y')$, and hence the set $vic\,y$ is finite.

Now, under the assumption that the covering $\mathcal{R}$ of S is star-finite, we define a partial ordering relation $\leqslant$ in Y, putting $y \leqslant y'$ iff $\Phi(y) \subsetneq \Phi(y')$. Then we consider the induced subgraph G' of G^{U}, whose vertex set is the set Y' of all maximal elements in $(Y, \leqslant)$.

6.5 *Proposition* The pretopological spaces $(G^{U}, P_{G^{U}})$ and $(G', P_{G'})$ belong to the same homotopy type.

Proof: For any $y \in Y$ there is some $y' \in Y'$ such that $y \leqslant y'$. In fact we have $\Phi(y') \subseteq \psi(j)$ for any $j \in \Phi(y)$. So we define a precontinuous map $f: G^{U} \to G'$, choosing for each $y \in Y$ an $y' \in Y'$ such that $y \leqslant y'$. Then we consider the canonical injection $g: G' \to G^{U}$. Clearly $fg = 1_{G'}$. Moreover we obtain a prehomotopy H of $1_{G^{U}}$ to gf, putting:

$$H(y,t) = \begin{cases} gf(y) & \text{for } t=1,\ y \in Y; \\ y & \text{for } t<1,\ y \in Y. \end{cases}$$

Remark. If the covering $\mathcal{R}$ of S is star-finite, the pf-spaces (S, R_{U}) and $(G', P_{G'})$ are homotopic by Propositions 6.4 and 6.5.

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