Nicolae Popa

Interpolation theorems for rearrangement invariant p-spaces of functions, $0 < p < 1$, and some applications

In: Zdeněk Frolík (ed.): Proceedings of the 10th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1982. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 2. pp. [199]--216.

Persistent URL: http://dml.cz/dmlcz/701275

INTERPOLATION THEOREMS FOR REARRANGEMENT INVARIANT p-SPACES OF FUNCTIONS, $0<p<1$, AND SOME APPLICATIONS

Nicolae Popa

In this paper we extend two interpolation theorems in the setting of rearrangement invariant p-spaces, for $0<p<1$.

Some applications of these theorems are given, particularly we extend Theorem 2.c.6 - [4] proving that the Haar system is an unconditional basis in a rearrangement invariant p-space X iff the Boyd indices p_X and q_X verify the relations $1<p_X$ and $q_X<\infty$. Some non locally convex Lorentz fonction spaces are examples of such rearrangement invariant p-spaces, while in [3] N.J.Kalton proved that only the locally convex Orlicz spaces have a Schauder basis.

In the sequel we assume all the vector spaces to be real. p is a positive real number less than 1.

Let X a topological complete vector space such that its topology is generated by a positive function $\| \ \|_X$, called p-norm, which fulfills the following properties: 1) $\|x\|_X = 0$ iff $x = 0$; 2) $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$ for $\alpha \in \mathbb{R}$, $x \in X$; 3) $\|x+y\|_X^p < \|x\|_X^p + \|y\|_X^p$ for $x, y \in X$. (We recall that $\| \ \|_X$ generates the topology of X if $U_n = \{x \in X; \|x\|_X \leqslant \frac{1}{n}\}$, $n \in \mathbb{N}$; constitute a neighbourhood basis of origin for this topology).

We say that X is a p-Banach space. If p = 1 we find the classical definition of a Banach space.

A p-Banach space $(X, \| \ \|)$ which is moreover a vector lattice, is called a p-Banach lattice if

$$|x| \leqslant |y| \text{ implies that } \|x\| \leqslant \|y\| \text{ for } x, y \in X.$$

We shall give the definition of a rearrangement invariant p-space of functions only in the case when the functions are defined on $I = [0,1]$. For more details about the rearrangement invariant p-spaces see [5].

A p-Banach space X of functions on I is called a p-<u>Köthe space of functions on</u> I if the following conditions are fullfilled.

a) X is a p-Banach lattice of μ-measurable functions on I with respect of pointwise order (μ is the Lebesgue measure). Moreover the functions of X are p-locally integrable.

b) If $f \in X$ and $g \in L_0(I)$ (the space of all Lebesgue measurable functions on I) such that $|g| \leq |f|$ μ-a.e., then it follows that $g \in X$ and $\|g\|_X \leq \|f\|_X$.

c) The characteristic function $\chi_A \in X$ for each $A \subset I$ such that $\mu(A) < \infty$.

d) The p-norm $\|f\|_X$ of X is p-<u>convex</u>, i.e. the μ-measurable function $\left(\sum_{i=1}^{n} |f_i|^p\right)^{1/p}$ belongs to X for $f_1, \ldots, f_n \in X$ and moreover

$$\left\| \left(\sum_{i=1}^{n} |f_i|^p\right)^{1/p} \right\|_X \leqslant \left(\sum_{i=1}^{n} \|f_i\|_X^p\right)^{1/p}$$

e) (Riesz-Fischer condition). If $f_1, \ldots, f_n, \ldots$ are elements of X and $\sum_{i=1}^{\infty} \|f_i\|_X^p < \infty$, then the μ-measurable function $\left(\sum_{i=1}^{\infty} |f_i(t)|^p\right)^{1/p}$ belongs to X.

The condition d) is very important and it is used to define a substitute of a "dual" for the rearrangement invariant p-space.

More precisely, let X be a p-Köthe space of functions on I. We denote by $X_{(p)}$ the set $\{x : I \longrightarrow \mathbb{R}$; such that the function $t \longrightarrow x(t)^{1/p} = |x(t)|^{1/p} \operatorname{sign} x(t)$ belongs to $X\}$.

Endowed with the usual operations, with the pointwise order and the norm $\|x\|_{(p)} = \left\| |x|^{1/p} \right\|_X^p$, $X_{(p)}$ becomes a Köthe space of functions on I, i.e. a 1-Köthe space of functions on I.

For instance if $X = L_p(0,1)$ then it follows that $X_{(p)} = L_1(0,1)$.

We can give also the dual construction.

Let X be a Köthe space of functions on I. We denote by $X^{(p)}$ the set $\{x : I \longrightarrow \mathbb{R}$; such that the function x^p belongs to $X\}$.

We consider for x $X^{(p)}$ the p-norm

$$\|x\|^{(p)} = \left\| |x|^p \right\|_X^{1/p}$$

Then $X^{(p)}$ becomes a p-Köthe space of functions on I with respect to usual operations and pointwise order.

For instance if $X = L_1(0,1)$ then it follows that $X^{(p)} = L_p(0,1)$.

If X is a p-Köthe space of functions on I then it is obvious that

$$X = \left[X_{(p)}\right]^{(p)} .$$

We can consider also the Köthe dual of $X_{(p)}$, $\left[X_{(p)}\right]' = \{g : I \longrightarrow \mathbb{R} ;$ $\int_0^1 |f(t)g(t)|\, dt < \infty$ for all $f \in X_{(p)}\}$. We introduce on $\left[X_{(p)}\right]'$ the norm

$$\|g\| = \sup_{\|f\|_{(p)} \leq 1} \int_0^1 |f(t)\, g(t)|\, dt$$

and $\left[X_{(p)}\right]'$ becomes a Köthe space of functions on I.

Then X is a vector sublattice of $X'' := \left\{ \left[X_{(p)}\right]'' \right\}^{(p)}$ but in general it is not a p-Banach subspace of it.

A p-Köthe space X of functions on I is called a rearrangement invariant p-space of functions (briefly r.i.p-space) in the following conditions hold.

1) For every $f \in X$ and every measure preserving automorphism $\sigma : I \longrightarrow I$ the function $f \circ \sigma$ belongs to X and moreover $\|f \circ \sigma\|_X = \|f\|_X$.

2) X is a p-Banach subspace of X'' and X is either <u>maximal</u> i.e. $X = X''$, or <u>minimal</u> i.e. the subspace of all simple p-integrable functions is dense in X.

3) We have the canonical inclusions

$$L_\infty(0,1) \subset X \subset L_p(0,1)$$

such that the norms of these maps are less than 1. (We denote by $\|T\|$ the expression $\sup\{\|Tx\| ;\ \|x\|_X \leq 1\}$, where $T : X \longrightarrow Y$ is a linear and bounded operator acting between the p-Banach spaces X and Y).

Interesting examples of r.i.p-spaces are p-Orlicz and p-Lorentz spaces.

Let $M : [0,\infty) \longrightarrow \mathbb{R}_+$ be a continuous, increasing and p-<u>convex</u> function. (We mention that a function $M : [0,\infty) \longrightarrow \mathbb{R}_+$ it is called p-<u>convex</u> if

$$M\left[(\alpha x^p + \beta y^p)^{1/p}\right] \leq \alpha M(x) + \beta M(y) \quad \text{for} \quad x,y \in \mathbb{R}_+ \text{ and}$$

$\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + \beta = 1$). If $M(0) = 0$, $M(1) = 1$ and if $\lim_{t \to \infty} M(t) = \infty$ we say that M is a p-<u>Orlicz function</u>.

Instead of an 1-Orlicz function we say simpler an <u>Orlicz function</u>.

The p-Orlicz space $L_M(0,1)$ is the space of all Lebesgue measurable functions $f : I \longrightarrow \mathbb{R}$ such that

$$\int_0^1 M\left(\frac{|f(t)|}{\rho}\right) dt < \infty$$

for some $\rho > 0$.

The p-norm on $L_M(0,1)$ is defined by

$$\|f\|_M = \inf\left\{\rho > 0;\ \int_0^1 M\left(\frac{|f(t)|}{\rho}\right) dt \leq 1\right\}.$$

It is not so difficult to prove that $L_M(0,1)$ is a r.i.-p-space maximal.

We mention also that, for $X = L_M(0,1)$, it follows that $X_{(p)} = L_{M_{(p)}}(0,1)$, where $M_{(p)}(t) = M(t^{1/p})$.

Of some interest is also the subspace $H_M(0,1) \subset L_M(0,1)$ of all Lebesgue measurable functions f defined on $[0,1]$ such that, for all $\rho > 0$, we have $\int_0^1 M\left(\frac{|f(t)|}{\rho}\right) dt < \infty$. $H_M(0,1)$ is a r.i.p-space minimal.

If $M(t) = \frac{e^{t^{2p}}-1}{e-1}$ then $H_M(0,1) \neq L_M(0,1)$.

Another interesting class of r.i.p-spaces is the class of p-Lorentz spaces.

Let $0 < q < \infty$ and let W be a continuous non-increasing positive function defined on $(0,\infty)$ such that $\lim_{t\to 0} W(t) = 0$,

$$\int_0^1 W(t)dt = 1 \text{ and } \int_0^\infty W(t)dt = \infty.$$

Let $0 < p \leq q < \infty$. Then the p-Lorentz space of functions $L_{W,q}(0,1)$ is the space of all Lebesgue measurable functions f on I such that

$$\|f\|_{W,q} = \left(\int_0^1 [f^*(t)]^q\, W(t)dt\right)^{1/q} < \infty$$

(Here is $f^*(t) = \inf_{\mu(E)=t} \sup_{s \notin E} |f(s)|$).

Then $L_{W,q}(0,1)$ is a r.i.p-space maximal, where $0 < p \leq 1$. We mention that, for $X = L_{W,q}(0,1)$, we have $X_{(p)} = L_{W,q/p}(0,1)$.

The r.i.p-spaces are used in interpolation theory. More precisely they constitute the natural framework for theorems of Calderon-Miteaghin and of Boyd.

In the sequel we present the extension of these theorems for r.i.p-spaces.

First of all we introduce an order relation on $L_p(0,1)$.

Let $f,g \in L_p(0,1)$, $0<p\leq 1$. We write $f \underset{p}{\prec} g$ if for all $s\in [0,1]$ we have

$$\int_0^s \left[f^*(t)\right]^p dt \leqslant \int_0^s \left[g^*(t)\right]^p dt$$

It is obvious that $f \underset{p}{\prec} g$ is equivalent to each of the following relations: $|f| \underset{p}{\prec} |g|$; $f^* \underset{p}{\prec} g^*$; $\lambda f \underset{p}{\prec} \lambda g$ for all real numbers $\lambda \neq 0$.

It is clear that $f \underset{p}{\prec} g$ and $g \underset{p}{\prec} h$ imply that $f \underset{p}{\prec} h$. Moreover $f \underset{p}{\prec} g$ and $g \underset{p}{\prec} f$ hold simultaneously if and only if $f^* = g^*$.

Another useful relation is the following

$$(f_1 \oplus f_2)^* \underset{p}{\prec} f_1^* \oplus f_2^*.$$

Here is $f_1 \oplus f_2 = (f_1^p + f_2^p)^{1/p}$.

It is true also a relation similarly to Riesz decomposition property, namely: Assume that $g \underset{p}{\prec} f_1 \oplus f_2$ for positive functions g, f_1, f_2. Then there exist the positive functions g_1, g_2 such that $g = g_1 \oplus g_2$ and $g_i \underset{p}{\prec} f_i$, $i = 1,2$.

Indeed $g^p \underset{1}{\prec} f_1^p + f_2^p$ and, by Proposition 2.a.7-[4], there exist $g_1', g_2' \geqslant 0$ in $L_1(0,1)$ such that $g_1' + g_2' = g^p$ and $g_i' \underset{p}{\prec} f_i^p$, $i=1,2$. We conclude denoting $(g_i')^{1/p}$ by g_i, $i=1,2$.

The next proposition shows us that a r.i.p-space X is an "ideal" for the order relation $\underset{p}{\prec}$. Namely

Proposition 1. Let X be a r.i.p-space on $[0,1]$. Assume that $g \underset{p}{\prec} f$ and $f\in X$. Then $g\in X$ and $\|g\|\leqslant\|f\|$.

Proof. The case $p = 1$ constitute Proposition 2.a.8-[4].

Let $0<p<1$. Then $g^p \underset{1}{\prec} f^p$ and, by the same Proposition it follows that $g^p \in X_{(p)}$ and $\|g\|_X^p = \|g^p\|_{(p)} \leqslant \|f^p\|_{(p)} = \|f\|_X^p$. ∎

An operator T from a p-Banach space X taking values into a p-Banach lattice Y is said to be quasilinear if :

1) $|T(\alpha x)| = |\alpha|\cdot|Tx|$ for all scalars α and $x\in X$.

2) There exists a constant $C<\infty$ such that

$$|T(x_1+x_2)| \leq C(|T x_1| + |T x_2|),\ x_1, x_2 \in X .$$

A quasilinear operator T is bounded if $\|T\|<\infty$.

Now we can state an extension of Calderon-Miteaghin's Theorem. (See Theorem 2.a.10- [4]).

Theorem 2. Let X be a r.i.p-space of functions on $[0,1]$.

Let T be a quasilinear operator define on $L_p(0,1)$, which is simultaneously bounded on $L_\infty(0,1)$ and $L_p(0,1)$.

Then T applies X into X and moreover

$$\|T\|_X \leq 2^{1/p-1} C \max (\|T\|_p, \|T\|_\infty),$$

where C is the constant aforementionned.

Proof. Let $f \in X$ and $0<s<1$.

Put $$g_s(t) = \begin{cases} f(t)-f^*(s) & \text{if } f(t)>f^*(s) \\ f(t)+f^*(s) & \text{if } f(t)<-f^*(s) \\ 0 & \text{if } |f(t)| \leq f^*(s) \end{cases}$$

and $h_s(t) = f(t) - g_s(t)$.

It is clear that $\|h_s\|_\infty = f^*(s)$ and, denoting by $A = \{t\in[0,1];\ f(t)>f^*(s)\}$, $B = \{t\in[0,1];\ f(t)<-f^*(s)\}$, we have $\mu(A\cup B) = \mu\{t\in[0,1];\ |f(t)|>f^*(s)\} := d_f(f^*(s)) \leq s$.

Hence

$$\|g_s\|_p^p + s\left[f^*(s)\right]^p = \int_0^1 \left[g_s(t)\right]^p dt + s\left[f^*(s)\right]^p =$$

$$= \int_A \{\left[f(t)-f^*(s)\right]^p + \left[f^*(s)\right]^p\}dt + \int_B \{\left[f^*(s)\right]^p + |f(t)+f^*(s)|^p\}dt +$$

$$(*) \quad +\left[s-\mu(A\cup B)\right]\cdot\left[f^*(s)\right]^p \leq 2^{1-p}\{\int_{A\cup B} |f(t)|^p + (s-\mu(A\cup B))\left[f^*(s)\right]^p\} \leq$$

$$\leq (\text{since } \int_0^s \left[f^*(t)\right]^p dt = \sup_{\mu(\sigma)=s} \int_\sigma |f(t)|^p dt) \leq$$

$$\leq 2^{1-p}\left[\int_0^{\mu(A\cup B)} \left[f^*(t)\right]^p dt + \int_{\mu(A\cup B)}^s \left[f^*(s)\right]^p dt\right] \leq 2^{1-p}\int_0^s \left[f^*(t)\right]^p dt.$$

Since $|Tf| \leq C(|Tg_s| + |Th_s|)$ we have

$$\int_0^s \left[(Tf)^*(t)\right]^p dt = \int_0^s \{\left[T(f)(t)\right]^p\}^* dt \leq (\text{since } f\leq g \text{ implies } f^*\leq g^*) \leq$$

$$\leq C^p \int_0^s \left[(|Tg_s| + |Th_s|)^p\right]^* dt \leq C^p \int_0^s (|Tg_s|^p + |Th_s|^p)^* dt \leq$$

$$\leq (\text{since } (f_1\oplus f_2)^* \underset{p}{\prec} f_1^*\oplus f_2^*) \leq C^p\left[\int_0^s (|Tg_s|^p)^* dt + \int_0^s (|Th_s|p)^* dt\right] \leq$$

$$\leq C^p \max(\|T\|_p^p, \|T\|_\infty^p)(\|g_s\|_p^p + s\,[f^*(s)]^p) \leqslant$$

$$\leqslant (\text{by } (*)) \leqslant 2^{1-p} C^p \max(\|T\|_p^p, \|T\|_\infty^p) \int_0^s [f^*(t)]^p\, dt.$$

Consequently $Tf \underset{p}{\prec} 2^{1/p-1} C \max(\|T\|_p, \|T\|_\infty)\cdot f$.

Hence, by Proposition 1, it follows that $Tf \in X$ and $\|Tf\| \leqslant$ $\leqslant 2^{1/p-1} C \max(\|T\|_p, \|T\|_\infty)\cdot \|f\|_X$. ∎

The natural projection $P_A(f) = f\chi_A$, where $A \subset [0,1]$ is a Lebesgue measurable subset and $f \in L_\infty(0,1)$, is the most common example of a simultaneously continuous operator on $L_p(0,1)$ and $L_\infty(0,1)$.

Another, more intricate example is given by $Tf(x) = \sum_{n=1}^{\infty} (n^{-3/p}) f(x^{1/n})$, where $f \in L_p(0,1)$ and $x \in [0,1]$.

Indeed Theorem 3.2-[2] shows us that, for every sequence $(a_n)_{n=1}^{\infty}$ of Borel functions on $[0,1]$ and for every sequence $(\sigma_n)_{n=1}^{\infty}$ of measurable functions on $[0,1]$ such that

$$(**) \qquad \sup_{\mu(B)>0} \frac{1}{\mu(B)} \sum_{n=1}^{\infty} \int_{\sigma_n^{-1}(B)} |a_n(x)|^p\, d\mu(x) = M < \infty ,$$

the expression $Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n(x))$, where $f \in L_p(0,1)$ and $x \in [0,1]$, defines a bounded operatur $T : L_p(0,1) \longrightarrow L_p(0,1)$ such that $\|T\| = M^{1/p}$.

If T has the aforementionned expression it is easy to prove the condition (**) for every Borel set B, consequently T is a continuous operator on $L_p(0,1)$.

Since $\|Tf\|_\infty \leqslant (\sum_{n=1}^{\infty} n^{-3/p}) \|f\|_\infty$ for $f \in L_\infty(0,1)$, it follows that T is a bounded operator on $L_\infty(0,1)$ too. Hence T applies X into X and it is bounded on it. (Here X is a r.i.p-space).

As an application of Theorem 2 we give the following example of a complemented subspace of a r.i.p-space of functions on $[0,1]$.

Corrolary 3. Let X be a r.i.p-space, $0<p<1$, and let $\sum_0$ be a σ-subalgebra of the σ-algebra $\mathcal{B}$ of all Borel subsets of $[0,1]$ containing the sets of Lebesgue measure equal to zero. If there exist $A \in \mathcal{B}$ and $\varepsilon > 0$ such that

$$(1) \qquad \mu(A \cap B) \geqslant \varepsilon \mu(B) \quad \text{for } B \in \textstyle\sum_0$$

and such that

(2) <u>for all Borel substes</u> $C \subset A$, <u>there exists</u> $B \in \sum_0$ <u>with</u> $B \cap A = C$, <u>then</u> $X(\sum_0) = \{f \in X;\ f$ <u>being</u> a $\sum_0$ -<u>measurable function</u>$\}$ <u>is a complemented subspace</u> of X.

<u>Proof</u>. Let P_A be the natural projection of $L_p(0,1)$ onto $L_p(A)$. By (1) it follows that the restriction of P_A on $L_p(\sum_0) = L_p((0,1), \sum_0, \mu)$ has a continuous inverse and (2) shows that P_A maps $L_p(\sum_0)$ onto $L_p(A)$. Hence $P_A|L_p(\sum_0) : L_p(\sum_0) \longrightarrow L_p(A)$ is a linear homeomorphism. Consequently $T = Q\,P_A$, where $Q = \left[P_A|L_p(\sum_0)\right]^{-1}$, is a continuous projection from $L_p(0,1)$ onto $L_p(\sum_0)$. Using (1) it follows that $\|P_A f\|_\infty = \|f\|_\infty$ for all $f \in L_\infty(\sum_0) = L_\infty((0,1), \sum_0, \mu)$ and by (2) we get that $P_A(L_\infty(\sum_0)) = L_\infty(A)$. Thus $T = Q\,P_A$ is a continuous projection from $L_\infty(0,1)$ onto $L_\infty(\sum_0)$. Applying Theorem 2 we get that T is a continuous projection from X into X. If $f \in X \subset L_p(0,1)$, then $Tf \in L_p(\sum_0) \cap X \subset X(\sum_0)$. Conversely, if $g \in X(\sum_0) \subset L_p(\sum_0)$, then $g = Tg$ and we are done. ∎

An example of a σ-algebra $\sum_0$ verifying the conditions (1) and (2) is the following.

$\sum_0 = \{B \cup C \cup D;\ B \subset [0,1/2]$ a Borel set, $C = \tau(B)$, where $\tau(x) = x + 1/2$ for $x \in [0, 1/2]$, and $\mu(D) = 0\}$.

Theorem 2 allows us to conclude that the linear operators simultaneously continuous on $L_\infty(0,1)$ and $L_p(0,1)$ act continuously on every r.i.p-space X. Since there exist interesting operators which are bounded only on some $L_q(0,1)$ with $p < q < \infty$, we shall study further the r.i.p-spaces X which are "between" $L_{p_1}(0,1)$ and $L_{p_2}(0,1)$, in the sense that every operator defined and bounded on these two spaces is defined and bounded also on X.

In this purpose we recall the definition of Boyd indices.

For $0 < s < \infty$ we define the operator D_s as follows.

For every measurable function f on $[0,1]$, put

$$(D_s f)(t) = \begin{cases} f(t/s) & t \leq \min(1,s) \\ 0 & s < t \leq 1. \end{cases}$$

Obviously $\|D_s\|_\infty \leq 1$ and

$$\|D_s\|_p^p = \sup_{\|f\|_p\leq 1} \|D_s f\|_p^p = \begin{cases} \sup\limits_{\|f\|_p\leq 1} \int_0^s |f(t/s)|^p\,dt = s & \text{for } s<1 \\ \sup\limits_{\|f\|_p\leq 1} \int_0^s |f(t/s)|^p\,dt \leq s & \text{for } s\geq 1. \end{cases}$$

Consequently $\|D_s\|_p = s^{1/p}$ and, by Theorem 2, it follows that D_s acts continuously on X and $\|D_s\|_X \leq 2^{1/p-1}\max(1,s^{1/p})$.

Moreover $(D_s f)^* \leq D_s f^*$ for every f and $0<s<\infty$. Consequently we can compute $\|D_s\|_X$ using only nonincreasing functions f. Since, for such a function f, we get $D_r f \leq D_s f$, where $0<r<s<\infty$, it is clear that $\|D_s\|_X$ is a nonincreasing function of s. Moreover $\|D_{rs}\|_X \leq \|D_r\|_X \cdot \|D_s\|_X$ for all $0<r,s<\infty$.

Now we can define the so-called Boyd indices p_X, q_X.

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|_X} = \sup_{s>1} \frac{\log s}{\log\|D_s\|_X},$$

$$q_X = \lim_{s\to 0^+} \frac{\log s}{\log\|D_s\|_X} = \sup_{0<s<1} \frac{\log s}{\log\|D_s\|_X}.$$

If $\|D_s\|_X = 1$ for some $s>1$ we put $p_X=\infty$. Similarly, if $\|D_s\|_X = 1$ for all $s<1$, we put $q_X = \infty$. Obviously $p_X = q_X = p$ for $X = L_p(0,1)$ where $0<p\leq\infty$.

Proposition 4. Let X be a r.i.p-space. Then

1) $p\leq p_X \leq q_X \leq \infty$.

2) $p_{X_{(p)}} = p_{X/p}$ and $q_{X_{(p)}} = q_{X/p}$.

Proof. 1) Since $\|D_s\|_X \leq 2^{1/p-1}\cdot s^{1/p}$ for $s\geq 1$, we get

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|_X} \geq \lim_{s\to\infty} \frac{\log s}{\log 2^{1/p-1}s^{1/p}} = p.$$

But $\|D_s\|_X \cdot \|D_{s^{-1}}\|_X \geq \|D_{ss^{-1}}\|_X = 1$, consequently

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|_X} \leq \lim_{s\to\infty} \frac{\log s^{-1}}{\log\|D_{s^{-1}}\|_X} = q_X.$$

2) Obviously $\|D_s\|_{X_{(p)}} = \|D_s\|_X^p$. ∎

Proposition 5. Let X be a r.i.p-space of functions on [0,1]. For every $p\leq p_1 < p_X$ and $q_X < q_1 \leq \infty$ we have $L_{q_1}(0,1)\subset X\subset L_{p_1}(0,1)$, the inclusion maps being continuous.

Proof. Proposition 2.b.3-[4] settles the case $p = 1$.

For $0<p<1$, by Proposition 4, we get $1\leqslant p_{1/p}<p_{X/p}=p_{X_{(p)}}$ and $q_{X_{(p)}}=q_{X/p}<q_{1/p}\leqslant\infty$. Applying again Proposition 2.b.3-[4] it follows that the inclusion maps $L_{q_{1/p}}\longrightarrow X_{(p)}$ and $X_{(p)}\longrightarrow L_{p_{1/p}}$ are continuous. Hence the inclusion maps $L_{q_1}\longrightarrow X$ and $X\longrightarrow L_{p_1}$ are also continuous. ∎

We recall the Theorem 2.b.6-[4] which will be useful in the sequel.

Theorem 6. Let X be a r.i. space. Then p_X (resp. q_X) is the minimum (resp. maximum) of all numbers p with the following property for every $\varepsilon>0$ and every integer n, X contains n disjoint functions $(f_i)_{i=1}^n$ equally distributed such that

$$(1-\varepsilon)\Big(\sum_{i=1}^n|a_i|^p\Big)^{1/p}\leqslant\Big\|\sum_{i=1}^n a_i f_i\Big\|_X\leqslant(1+\varepsilon)\Big(\sum_{i=1}^n|a_i|^p\Big)^{1/p}$$

for every choice of scalars $(a_i)_{i=1}^n$.

We give further another interpolation theorem which extends the Boyd interpolation theorem.

First of all we introduce the spaces $L_{r,q}$, whre $p\leqslant r,q\leqslant\infty$. For $p\leqslant r\leqslant\infty$ and $p\leqslant q<\infty$ we denote by $L_{r,q}(0,1)$ the space of all Lebesgue measurable functions on $[0,1]$ such that

$$\|f_{r,q}\|=\Big[q/r\int_0^1\big[t^{1/r}f^*(t)\big]^q\frac{dt}{t}\Big]^{1/q}<\infty.$$

For $p\leqslant r\leqslant\infty$ we denote by $L_{r,\infty}(0,1)$ the space of all Lebesgue measurable function f such that

$$\|f\|_{r,\infty}=\sup_{t>0}t^{1/r}f^*(t)<\infty.$$

(For more details about the spaces $L_{p,q}$ see [1]).

Obviously $L_{q,q}(0,1)=L_q(0,1)$ and $\|f\|_{q,q}=\|f\|_q$. Moreover we have $\|f\|_{r,q_2}\leqslant\|f\|_{r,q_1}$ for $0<q_1\leqslant q_2\leqslant\infty$ [1], thus

$$L_{r,q_1}(0,1)\subset L_{r,q_2}(0,1).$$

By Holder's inequality we get:

$$L_{r_3,\infty}(0,1)\subset L_{r_2,q_1}(0,1)\subset L_{r_1,q_2}(0,1)$$

where $0<r_1<r_2<r_3\leqslant\infty$ and $q_1,q_2>0$.

The spaces $L_{r,q}(0,1)$ are topologically complete metrizable vector spaces. (See [1]).

If $p \leq q < r$, then the space $L_{r,q}(0,1)$ coincides with the p-Lorentz space $L_{W,q}(0,1)$, where $W(t) = \frac{q}{r} \cdot t^{q/r-1}$, $0 < t < \infty$.

It is interesting to mention that $L_{p,\infty}(0,1)$ cannot be p-renormed such that the p-norm be p-convex.

Let now $p \leq r_1 \leq \infty$ and let T be a linear map defined on a subset of $L_{r_1}(0,1)$ with values in $L_0(0,1)$.

1) The map T is said to be of <u>strong type</u> (r_1,r_2) for a suitable $r_2 \in [p,\infty]$, if there exists a constant $M>0$ such that $\|Tf\|_{r_2} \leq M\|f\|_{r_1}$ for every f from the domain of definition of T.

2) T is said to be of <u>weak type</u> (r_1,r_2) for some $r_2 \in [p,\infty]$ if there exists a constant $M>0$ such that

$$\|Tf\|_{r_2,\infty} \leq M\|f\|_{r_1,p}$$

for every f from the domain of definition of T. We make the convention that, for $r_1 = \infty$, instead of $\|f\|_{\infty,p}$ we put $\|f\|_{\infty,\infty} = \|f\|_{\infty}$.

It is clear that an operator of strong type (r_1,r_2) is also of weak type (r_1,r_2). Finally we remark that T is of weak type (r_1,r_2) if and only if there exists a constant $M>0$ such that

$$\sup_{t>0} t\cdot(\mu\{s\in [0,1] ; |Tf(s)| \geq t\})^{1/r_2} \leq M(p/r_1 \int_0^1 t^{p/r_1-1} [f^*(t)]^p dt)^{1/p} .$$

We prove now the extension of Theorem 2.b.11-[4].

<u>Theorem 7. Let</u> $0<p\leq 1$ <u>and</u> $p \leq p_1 < q_1 \leq \infty$ <u>and let</u> T <u>be a linear operator acting from</u> $L_{p_1,p}(0,1)$ <u>into</u> $L_0(0,1)$.

<u>Assume that</u> T <u>is of weak types</u> (p_1,p_2) <u>and</u> (q_2,q_1). <u>Then for every r.i.p-space</u> X <u>of functions on</u> $[0,1]$ <u>such that</u> $p_1 < p_X$ <u>and</u> $q_X < q_1$, T <u>maps into itself and it is bounded on</u> X.

The following lemma is an extension of Lemma 2.b.12-[4].

<u>Lema 8. With the same assumptions on</u> T <u>as in Theorem</u> 7 <u>there is a constant</u> $M<\infty$ <u>such that</u>

$$[(Tf)^*(2t)]^p \leq M \left[\int_0^1 [f^*(tu)]^p u^{p/p_1-1} du + \int_1^\infty [f^*(tu)]^p u^{p/q_1-1} du \right]$$

<u>for every</u> $0<t\leq 1/2$ <u>and</u> $f\in L_{p_1,p}(0,1)$.

<u>Proof</u>. Suppose that T is of weak types (p_1,p_1) and (q_1,q_1) with the constants M_{p_1} and M_{q_1}. Let $f\in L_{p_1,p}(0,1)$ and for $u,t\in [0,1]$ set

$$g_t(u) = \begin{cases} f(u) - f^*(t) & \text{if } f(u) > f^*(t) \\ f(u) + f^*(t) & \text{if } f(u) < -f^*(t) \\ 0 & \text{if } |f(u)| \leq f^*(t) \end{cases}$$

and $h_t(u) = f(u) - g_t(u)$.

It is clear that $g_t, h_t \in L_{p_1,p}(0,1)$ and we apply the fact that T is of weak type (p_1,p_1) to g_t and of weak type (q_1,q_1) to h_t. Note that $g_t^*(u) = 0$ for $u \in [t,\infty)$ and $g_t^*(u) \leq f^*(u)$ for $0<u<t$. Hence, for $t \in I$, we have

$$t^{p/p_1}\left[(Tg_t)^*(t)\right]^p \leq M_{p_1}^p (p/p_1) \int_0^\infty \left[g_t^*(s)\right]^p s^{p/p_1} ds \leq$$

$$\leq M_{p_1}^p (p/p_1) \int_0^t \left[f^*(s)\right]^p s^{p/p_1-1} ds = M_{p_1}^p \left(\frac{p}{p_1}\right) t^{p/p_1} \int_0^1 \left[f^*(tu)\right]^p u^{p/p_1-1} du.$$

Since $|h_t(u)| = \min(|f(u)|, f^*(t))$, for $t \in [0,1]$, we have

$$t^{p/q_1}\left[(Th_t)^*(t)\right]^p \leq M_{q_1}^p \frac{p}{q_1} \int_0^\infty \left[h_t^*(s)\right]^p s^{p/q_1-1} ds \leq$$

$$\leq M_{q_1}^p \cdot \frac{p}{q_1} \cdot \left(\int_0^t \left[f^*(t)\right]^p s^{p/q_1-1} ds + \int_t^\infty \left[h_t^*(s)\right]^p s^{p/q_1-1} ds\right) =$$

$$= M_{q_1}^p \cdot \frac{p}{q_1} \cdot \left(\frac{q_1}{p} \left[f^*(t)\right]^p \cdot t^{p/q_1} + t^{p/q_1} \int_1^\infty \left[h_t^*(tu)\right]^p u^{p/q_1-1} du\right) \leq$$

$$\leq M_{q_1}^p \cdot \frac{p}{q_1} t^{p/q_1} \left(\frac{q_1}{p} \int_0^1 \left[f^*(tu)\right]^p u^{p/p_1-1} du + \int_1^\infty \left[f^*(tu)\right]^p u^{p/q_1-1} du\right).$$

Since $|Tf| \leq |Tg_t| + |Th_t|$ it follows that

$$\left[(Tf)^*(2t)\right]^p \leq \left[(Tg_t)^*(t) + (Th_t)^*(t)\right]^p \leq \left[(Tg_t)^*(t)\right]^p + \left[(Th_t)^*(t)\right]^p \leq$$

$$\leq \left(M_{p_1}^p \frac{p}{p_1} + M_{q_1}^p\right) \int_0^1 \left[f^*(tu)\right]^p u^{p/p_1-1} du + M_{q_1}^p \frac{p}{q_1} \int_1^\infty \left[f^*(tu)\right]^p u^{p/q_1-1} du.$$

This proves our lemma with $M = \frac{p}{p_1} M_{p_1}^p + M_{q_1}^p$. ∎

Proof of Theorem 7. Let p_o and q_o be such that $p_1 < p_o < p_X$ and $q_X < q_o < q_1$. Then there is $s_o > 1$ such that, for $s \geq s_o$, we have $p_o < \frac{\log s}{\log\|D_s\|_X}$. Consequently $\|D_s\|_X \leq s^{1/p_o}$ for $s \geq s_o$.

Since $s \longrightarrow \frac{\log s}{\log \|D_s\|_X}$ is an increasing function on $(1,\infty)$, it follows that there is $K < \infty$ such that $\|D_s\|_X \leq K s^{1/p_o}$ for $2 \leq s \leq \infty$.

Similarly, we can assume that $\|D_s\|_X \leq K s^{1/q_0}$ for $0 < s \leq 2$.

Let now $g \in X' = [X_{(p)}]'$ such that $\|g\|_{X'} = 1$ and put on

$$\tilde{g}(t) = \begin{cases} g(t) & \text{if } t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}.$$

Then we get

$$\int_0^1 \left(\int_0^1 [f^*(tu/2)]^p g(t) u^{p/p_1 - 1} du \right) dt = \int_0^1 u^{p/p_1-1} \left(\int_0^\infty (D_{2/u} f^*)^p(t) g(t) dt \right) du \leq$$

$$\leq \int_0^1 \|(D_{2/u} f^*)^p\|_{X_{(p)}} \cdot u^{p/p_1-1} du \leq K^p \cdot 2^{p/p_0} \left(\int_0^1 u^{p/p_1 - p/p_0 - 1} du \right) \|f\|_X^p =$$

$$= 2^{p/p_0} K^p \left(\frac{p}{p_1} - \frac{p}{p_0}\right)^{-1} \|f\|_X^p$$ for $f \in X$. Moreover, for $0 < t \leq \frac{1}{2}$,

$$\int_0^1 \left(\int_1^\infty [f^*(tu/2)]^p g(t) u^{p/q_1 - 1} du \right) dt = \int_1^\infty u^{p/q_1-1} \left(\int_0^1 (D_{2/u} f^*)^p(t) g(t) dt \right) du \leq$$

$$\leq \int_1^\infty u^{p/q_1 - 1} \|D_{2/u}\|_X^p \cdot \|f\|_X^p du \leq \|f\|_X^p K^p 2^{p/p_0} \int_1^\infty u^{p/q_1 - p/q_0 - 1} du =$$

$$= \|f\|_X^p \cdot 2^{p/q_0} K^p \left(\frac{p}{q_0} - \frac{p}{q_1}\right)^{-1}.$$

By Lemma 8 it follows that $\int_0^1 [(Tf)^*(t)]^p f(t) dt \leq M_0 \|f\|_X^p$ for $g \in X'$ such that $\|g\|_{X'} = 1$. Here $M_0 = MK^p \left(\frac{p}{p_1} - \frac{p}{p_0}\right)^{-1} 2^{p/p_0} + 2^{p/q_0} \left(\frac{p}{q_0} - \frac{p}{q_1}\right)^{-1}$, M being the constant appearing in Lemma 8.

Hence $(Tf)^p \in [X_{(p)}]''$. In other words $Tf \in \{[X_{(p)}]''\}^{(p)} = X''$. Moreover $\|Tf\|_{X''}^p = \|(Tf)^p\|_{X''_{(p)}} \leq M_0 \|f\|_X^p$.

If X is maximal, then $Tf \in X$ and $\|Tf\|_X \leq M_0 \|f\|_X$. Since $L_{q_0}(0,1)$ is a maximal r.i.p-space, then it follows as above that $T(L_{q_0}(0,1)) \subset \subset L_{q_0}(0,1)$. X being the closure of $L_{q_0}(0,1)$ for the topology of X'' it follows that T maps X into X and it is bounded there. ∎

Since $p_X = q_X = r > 1$, when $X = L_{r,p}(0,1)$ where $0 < p \leq 1 < r < \infty$, we get a r.i.p-space X non locally convex such that $1 < p_X \leq q_X < \infty$.

We shall give an application of Theorem 7.

Let $\mathcal{A}$ be a σ-subalgebra of $\mathcal{B}$ (the σ-algebra of all Borel subsets of $I = [0,1]$) such that the Lebesgue measure restricted on $\mathcal{A}$ is σ-finite. For $f \in L_1(0,1)$, the Lebesgue-Nikodym theorem shows the existence of a unique $\mathcal{A}$-measurable and Lebesgue integrable function, denoted by $E^{\mathcal{A}}f$, which verifies the relation

$$\int_0^1 (E^{\mathcal{A}}f)g\,dt = \int_0^1 gf\,dt$$

for every bounded $\mathcal{A}$-measurable function g on $[0,1]$.

It is clear that $f \longrightarrow E^{\mathcal{A}}f$ is an idempotent operator. This operator is called the <u>conditional expectation</u> and has the norm one on $L_1(0,1)$ and $L_\infty(0,1)$. Thus the norm of $E^{\mathcal{A}}$ is equal to 1 on $L_q(0,1)$ for all $1 \leq q \leq \infty$.

<u>Corollary 9.</u> <u>With the notations of above, if</u> $0<p\leq 1\leq p_1<q_1\leq\infty$ <u>and if X is a r.i.p-space of functions on</u> $[0,1]$ <u>such that</u> $p_1<p_X\leq$ $\leq q_X<q_1$, <u>then</u> $E^{\mathcal{A}}$ <u>maps</u> X <u>into itself and it is bounded on it.</u>

<u>Proof</u>. Since $p_1 \geq 1$ then $E^{\mathcal{A}}$ is an operator of strong types(p_1,p_1) and (q_1,q_1). Thus by Theorem 7 $E^{\mathcal{A}}$ maps X into itself and its norm does not depend on $\mathcal{A}$. ∎

Now we give an interesting application of Corrolary 9. Recall that the Haar system $(\chi_n)_{n=1}^{\infty}$ is given by $\chi_1(t)\equiv 1$ and, for $\ell=1,2,\dots,2^k$ and $k = 0,1,\dots$, by

$$\chi_{2^k+1}(t) = \begin{cases} 1 & \text{for } t\in [(2\ell-2)2^{-k-1},\ (2\ell-1)2^{-k-1}) \\ -1 & \text{for } t\in [(2\ell-1)2^{-k-1},\ (2\ell.2^{-k-1}) \\ 0 & \text{otherwise.} \end{cases}$$

N.J.Kalton showed in [3] that in a p-Orlicz space X the Haar system is a Schauder basis (i.e. every $f\in X$ admits a unique expansion $f = \sum_{i=1}^{\infty} a_i\chi_i$, where $(a_i)_{i=1}^{\infty}$ is a sequence of scalars and the sum converges for the topology of X) if and only if X is locally convex.

Particularly, the Haar system $(\chi_n)_{n=1}^{\infty}$ is not a Schauder basis in $L_p(0,1)$ for $0<p<1$. (See [6]).

Thus it is natural to ask whenever the Haar system is a Schauder basis in a r.i.p-space, for $0<p<1$. In order to answer to this question we associate to the Haar system an increasing sequence of σ-algebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of Lebesgue measurable subsets of $[0,1]$. σ-algebra $\mathcal{A}_1$ consist of the vanishing set $\emptyset$ and $[0,1]$. For $n = 2^k+\ell$, $1\leq \ell\leq 2^k$, $k\geq 0$, $\mathcal{A}_n$ is the σ-algebra spanned by $\mathcal{A}_{n-1}$ and the intervals $[(2\ell-2)2^{-k-1},\ (2\ell-1)2^{-k-1})$, $[(2\ell-1)2^{-k-1},\ 2\ell\cdot 2^{-k-1})$. It is clear that $\mathcal{A}_n$ is the smallest σ-algebra $\mathcal{A}$ such that the function $\{\chi_1,\dots,\chi_n\}$ are $\mathcal{A}$-measurables.

We can now prove the following assertion.

<u>Corollary 10.</u> <u>If</u> X <u>is a separable r.i.</u>p-<u>space of functions on</u>

$[0,1]$ <u>such that</u> $0<p<1\leqslant p_1<p_X\leqslant q_X<q_1\leqslant\infty$, <u>then the Haar system</u> $(\chi_n)_{n=1}^{\infty}$ <u>is a Schauder basis of</u> X.

<u>Proof</u>. Since X is not isomorphic to $L_\infty(0,1)$ then $\lim_{t\to 0}\|\chi_{(0,t)}\|_X = 0$. Consequently every simple function on $[0,1]$ can be approximated in the norm of X by the characteristic functions of dyadic intervals $2^{-k},(\ +1)2^{-k}),\ 0\qquad 2^k-1,\ k=0,1,\dots$

It follows that the Haar system spans a dense subspace in X.

Observe also that for n m and for every choice of scalars $a_i\ {}_{i=1}^{n}$ we have

$$E^{A_n}\left(\sum_{i=1}^{m} a_i\chi_i\right) = \sum_{i=1}^{n} a_i\chi_i$$

and, by Corrolary 9, it follows that $\|E^{A_n}\|_X\leqslant M$ for all $n\in\mathbb{N}$. Thus $(\chi_i)_{i=1}^{n}$ is a basic sequence in X. (see Theorem III 2.12-[6]). ∎

<u>Remark 11</u>. The restriction imposed in Corrolary 10 that $1<p_X\leqslant$ $\leqslant q_X<\infty$ is necessary, since in the case $p_X\leqslant 1$ or $q_X=\infty$ Corrolary 10 is not merely true.

For instance it is known (see [1]) that $L_{r,q}(0,1)$, where $0<r<1$, $0<q<\infty$ and $L_{1,q}(0,1)$ for $1<q<\infty$, are r.i.p-spaces X, where $0<p<$ $<r<1$, such that $X^* = \{0\}$. Moreover $p_X = q_X\leqslant 1$.

View of Remark 11 it is natural to ask following question.

<u>Problem 12</u>. Does there exist a separable non locally convex r.i. -space X such that $p_X = q_X = 1$ having a Schauder basis ?

It is clear that in $L_{r,q}(0,1)$, where $0<r<1<q<\infty$, the Haar system is a Schauder basis and however $L_{r,q}(0,1)$ is not locally convex.

We are further interested to know whenever the Haar system is an unconditional basis in a r.i.p-space of functions on $[0,1]$. We recall that a Schauder basis in X is an <u>unconditional basis</u> if the expansion of every element of X with respect to this basis converges unconditionally.

It is interesting to remark that the relation $1<p_X\leqslant q_X<\infty$ is a necessary and sufficient condition for the unconditionality of the basis $(\chi_n)_{n=1}^{\infty}$ in every r.i.p-space X. We extend in this way Theorem 2.c.6-[4].

<u>Theorem 13</u>. <u>Let</u> X <u>be a separable r.i.p-space of functions on</u> $[0,1]$. <u>The Haar system</u> $(\chi_n)_{n=1}^{\infty}$ <u>is an unconditional basis in X if and only if</u> $1<p_X\leqslant q_X<\infty$.

<u>Proof</u>. If $1<p_X\leqslant q_X<\infty$ then by Theorem 7 and using the fact that the Haar system is an unconditional basis in $L_q(0,1)$ for all

$1<q<\infty$ (see Theorem 2.c.5-[4]), we get that the projections P_σ from X into the subspace $[\chi_i]_{i\in\sigma}\subset X$, where $\sigma\subset \mathbb{N}$ is a closed subset, are uniformly bounded. Thus $(\chi_i)_{i=1}^{\infty}$ is an unconditional basis in X.

Conversely, assume that $(\chi_i)_{i=1}^{\infty}$ is an unconditional basis in X. By Proposition 4, $p_{X_{(p)}} = p_{X/p}$; consequently Theorem 6 shows that $\ell_{p_{X_{(p)}}}(n)$ spanned by positive disjoint elements having the same distribution function are uniformly contained in $X_{(p)}$. It follows that X contains uniformly the spaces $\ell_{p_X}(n)$ spanned by positive disjoint functions having the same distribution function.

In other words there is $M>0$ such that for all $n\in\mathbb{N}$ there are 2^n disjoint functions $(u_i)_{i=1}^{2^n}$ having the same distribution function, such that $\|u_i\|_X = 1$ and verifying the inequality

$$(*)\qquad M\Big(\sum_{i=1}^{2^n}\|u_i\|_X^{p_X}\Big)^{1/p_X} \geqslant \Big\|\sum_{i=1}^{2^n} u_i\Big\|_X \geqslant M^{-1}\Big(\sum_{i=1}^{2^n}\|u_i\|_X^{p_X}\Big)^{1/p_X}.$$

Let $(h_i)_{i=1}^{2^n}$, the Haar system over $(u_i)_{i=1}^{2^n}$, defined by

$$h_1 = 2^{-n/p_X}(u_1+\dots+u_{2^n})$$

$$h_2 = 2^{-n/p_X}(u_1+\dots+u_{2^{n-1}}-u_{2^{n-1}+1}-\dots-u_{2^n})$$

$$\vdots$$

$$h_{2^{n-1}+1} = 2^{-n/p_X}(u_1-u_2)$$

$$\vdots$$

$$h_{2^n} = 2^{-n/p_X}(u_{2^n-1}-u_{2^n}).$$

Since X is separable we can assume that u_i is a finite linear combination of characteristic functions of intervals $((\ell_j-1)2^{-k},\ \ell_j\cdot 2^{-k})$ for some k non depending of i. Applying a suitable automorphism of $[0,1]$ we can suppose that on the first 2^n dyadic intervals of length 2^{-k} every u_i is non-zero exactly on some of those intervals and takes there a value nondepending of i, say β_1. The same fact is also true for the following 2^n dyadic intervals of length 2^{-k}, where β_1 is replaced by β_2 and so on.

Thus, for some $m\in\mathbb{N}$ and some scalars $(\beta_j)_{j=1}^{m}$ we have

$$u_i = \sum_{j=1}^{m}(\beta_j\chi_{[(i-1+(j-1)2^n)2^{-k},\ (i+(j-1)2^n)2^{-k})},\qquad 1\leq i\leq 2^n.$$

Remark that

$$2^{n/p_X} h_2 = u_1 + \ldots + u_{2^{n-1}} - u_{2^{n-1}+1} - \ldots - u_{2^n} = \sum_{j=1}^{m} \beta_j \chi_{2^{k-n}+j}$$

$$2^{n/p_X} h_3 = u_1 + \ldots + u_{2^{n-2}} - u_{2^{n-2}+1} - \ldots - u_{2^{n-1}} = \sum_{j=1}^{m} \beta_j \chi_{2^{k-n+1}+2j-1}$$

$$2^{n/p_X} h_4 = u_{2^{n-1}+1} + \ldots + u_{2^{n-1}+2^{n-2}} - u_{2^{n-1}+2^{n-2}+1} - \ldots - u_{2^n} =$$

$$= \sum_{j=1}^{m} \beta_j \chi_{2^{k-n+1}+2j},$$

and so on.

In other words $\{h_j\}_{j=2}^{2^n}$ constitutes a block basis for a permutation π of the Haar basis $(\chi_n)_{n=1}^{\infty}$ of X. Thus the unconditionality constant K_n of $\{h_j\}_{j=2}^{2^n}$ (K_n is equal by definition, to $\sup\{\|\sum_{i=2}^{2^n} a_i\theta_i h_i\|_X;\ \|\sum_{i=2}^{2^n} a_i h_i\|_X \leq 1;\ \theta_i = \pm 1\}$) is less than K_X, the unconditionality constant of the basis $(\chi_n)_{n=1}^{\infty}$ of X.

Let now $T_n : [u_i]_{i=1}^{2^n} \longrightarrow \ell_{p_X}(2^n)$ given by $T_n(u_i) = e_i$, $1 \leq i \leq 2^n$, be an isomorphism which (by (*)) satisfies the relation

$$\|T_n\| \cdot \|T_n^{-1}\| \leq M^2 \quad \text{for all } n \in \mathbb{N}.$$

If $S_n : \ell_{p_X}(2^n) \longrightarrow L_{p_X}(0,1)$ is the isometry given by $S_n(e_i) = 2^{n/p_X} \chi_{[(i-1)2^{-n},\ i\,2^{-n})}$, then $U_n = S_n \circ T_n$ verifies the condition

$$\|U_n\| \cdot \|U_n^{-1}\| \leq M^2 \qquad n = 1,2,\ldots$$

and moreover we get

$$U_n(h_i) = \chi_i$$

for $1 \leq i \leq 2^n$, $n=1,2,\ldots$.

Thus the unconditionality constant of the system $(h_i)_{i=1}^{2^n}$ is the same, up to a factor M^2, as that of first 2^n elements of Haar system in $L_{p_X}(0,1)$. If $p_X \leq 1$, since the Haar system is not an unconditional basis in $L_{p_X}(0,1)$, then it follows that $K_n \xrightarrow[n]{} \infty$. Consequently $K_X = \infty$ which contradicts the fact that $(\chi_n)_{n=1}^{\infty}$ is an unconditional basis in X. Thus $1 < p_X$ and similarly we can prove that $q_X < \infty$. ∎

Consequently the Haar system is an unconditional basis in $L_{r,q}(0,1)$, where $0<q<1<r<\infty$, in spite of the fact that this space is not locally-convex.

REFERENCES

[1] - HUNT R. - On L(p,q) spaces. L'enseign. math. 12, (1966), 249-274.

[2] - KALTON N.J. - The endomorphisms of L_p $(0\leq p\leq 1)$. Indiana Univ. J. 27, (1978), 353-381.

[3] - KALTON N.J. - Compact and strictly singular operators on Orlicz spaces. Israel J.Math.26, (1977), 126-136.

[4] - LINDENSTRAUSS J., TZAFRIRI L. - Classical Banach spaces II, Springer Verlag, Berlin 1979.

[5] - POPA N. - Uniqueness of the symmetric structure in $L_p(\mu)$ for $0<p<1$ - to appear in Rev.Roum.Math.Pures Appl. (1982).

[6] - ROLEWICZ S. - Metric linear spaces, PWN, Warszawa, 1972.

DEPARTMENT OF MATHEMATICS
INCREST - BUCHAREST 79622
ROMANIA