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CANONICAL ORDERING THEOREMS , A FIRST ATTEMPT

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§ 1 Introduction

In this paper we investigate canonization theorems for total orders, these form the counterpart to 'canonical partition theorems' (see e.g. [4]) generalizing the notion of ordering theorems (see e.g. [3]).
It proves to be convenient to use the language of categories in order to define the general concept.

Let $\mathbb{C}$ be a category. For the applications $\mathbb{C}$ will always satisfy certain additonal properties, namely $\mathbb{C}$ is rigid, sceletal and every morphism is a monomorphism. For objects A and B the binomial coefficient $\mathbb{C}\binom{A}{B}$ denotes the set of morphisms (subobjects) $f : B \to A$.

Notation: ORD $\mathbb{C}\binom{A}{C}$ denotes the set of total orders on $\mathbb{C}\binom{A}{C}$.

Definition: A set $\Omega \subseteq$ ORD $\mathbb{C}\binom{B}{C}$ is a canonizing (by abuse of language also 'canonical') set ot total orders for $\mathbb{C}\binom{B}{C}$ iff Ω is a minimal set (with respect to inclusion) satisfying:
(ORD) there exists an object A in $\mathbb{C}$ such that for every total order $\leq \in$ ORD $\mathbb{C}\binom{A}{C}$ there exists a B-subobject $f \in \mathbb{C}\binom{A}{B}$ such that $\leq_f \in \Omega$, where $g \leq_f h$ iff $f \cdot g \leq f \cdot h$.

§ 2 Results

(2.1) Affine points in finite affine spaces

Let F be a finite field. Let Aff_F be a category which has as objects the affine spaces F^k , where k is a nonnegative integer. For $k \leq n$ let the morphisms $f \in \mathrm{Aff}_F\binom{n}{k}$ correspond bijectively to k-dimensional affine subspaces of F^n .

Particularly $\mathrm{Aff}_F\binom{n}{0}$ can be identified with F^n viewed as column vectors $(x_0,\ldots,x_{n-1})^T$. Analogously $\mathrm{Aff}_F\binom{n}{1}$ can be identified with the set of $n \times 2$ matrices such that there exists an index $i < n$ satisfying $y_\nu = 0$ for all $\nu < i$, $y_i = 1$ and $x_i = 0$. As usual A describes the line $\{(x_0,\ldots,x_{n-1})^T + \lambda \cdot (y_0,\ldots,y_{n-1})^T \mid \lambda \in F\}$.

For a total order $\leq \in \mathrm{ORD}(F)$ we denote by $\leq^* \in \mathrm{ORD}(F^m)$ the lexicographic order on F^m coming from $\leq$, i.e. $(x_0,\ldots,x_{m-1})^T \leq^* (y_0,\ldots,y_{m-1})^T$ iff there exists an index $i < m$ such that $x_\nu = y_\nu$ for all $\nu < i$ and $x_i < y_i$.

Theorem 1 The set $\Omega = \{\leq^* \in \mathrm{ORD}\ (F^m) \mid \leq \in \mathrm{ORD}\ (F)\}$ is a canonical set of total orders for $\mathrm{Aff}_F\binom{m}{0}$.

Proof: We verify the property (ORD) . According to the Graham-Leeb-Rothschild partition theorem [1] for finite affine spaces we can assume that $\lesssim \in \mathrm{ORD}\ \mathrm{Aff}_F\binom{m}{0}$ is given in such a way that each two affine lines of F^m are ordered of the same type. This gives an order $\leq \in \mathrm{ORD}\ (F)$. But then $\leq^* = \lesssim$, because if $\hat{x} = (\hat{x}_0,\ldots,\hat{x}_{m-1})^T$ and $\hat{y} = (\hat{y}_0,\ldots,\hat{y}_{m-1})^T$ are two different elements of F^m , then let $A \in \mathrm{Aff}_F\binom{m}{1}$ describe the affine line containing $\hat{x}$ and $\hat{y}$. Say that $\hat{x} = (x_0,\ldots,x_{m-1})^T + \lambda \cdot (y_0,\ldots,y_{m-1})^T$ and $\hat{y} = (x_0,\ldots,x_{m-1}) + \mu \cdot (y_0,\ldots,y_{m-1})^T$, where λ and μ are elements of F . Hence $\hat{x} \lesssim \hat{y}$ iff $\lambda \leq \mu$ which shows that $\lesssim$ is the lexicographic order coming from $\leq$.
The minimality of Ω is obvious, in fact Ω is uniquely determined.

(2.2) Points in Boolean algebras

Let $\mathcal{B}$ be a category which has objects the Boolean algebras B(k) , where k is a nonnegative integer. For $k \leq n$ let the morphisms $f \in \mathcal{B}\binom{n}{k}$ correspond bijectively to B(k) - subalgebras of B(n) . B(k) consists of all 0-1 sequences of length k ordered by the product order taken over $(2,\leq)$, viz. $0 < 1$.

A $B(k)$ - subalgebra of $B(n)$ can be represented by a 0-1 sequence $\hat{x} = (x_0,\ldots,x_{n-1})^T$, yielding the minimum of the subalgebra, and by k mutually disjoint and nonempty sets $I_1,\ldots,I_k$ which are subsets of $n = \{0,\ldots,n-1\}$ such that $x_\nu = 0$ for every $\nu \in I_1 \cup\ldots\cup I_k$. The representation is rigid if we require additionally that $\min I_1 < \min I_2 <\ldots< \min I_k$. The 0-1 sequence $y^i = (y_0^i,\ldots,y_{n-1}^i)^T$, where $y_0^i = x_\nu$ for $\nu \notin I_1 \cup\ldots\cup I_k$, $y_\nu^i = 1$ for $\nu \in I_i$ and $y_\nu^i = 0$ else yields the i.th atom of the $B(k)$ - subalgebras.

Recall that a $B(k)$ - subalgebra of $B(n)$ may be interpreted particularly as a k-dimensional affine subspace of $(GF(2))^n$, but generally not vice versa. However, essentially the same result as stated in theorem 1 for $GF(2)$ is valid for $\mathcal{B}$:

Theorem 2 The set $\Omega = \{\leq^*,\leq^{**}\} \subseteq \mathrm{ORD}\,(\mathcal{B}\binom{m}{0})$, where $\leq^*$ is the lexicographic order coming from $0 < 1$ and $\leq^{**}$ is the lexicographic order coming from $1 < 0$, is a canonical set of total orders for $\mathcal{B}\binom{m}{0}$.

Proof: We verify the property (ORD). According to the Graham-Rothschild partition theorem [2] for finite Boolean algebras we can assume that $\leq \in \mathrm{ORD}\,(\mathcal{B}\binom{m}{0})$ is given in such a way that each two $B(1)$ - sublattices and also each two $B(2)$-sublattices are ordered of the same type. We can also assume that $m \geq 3$.
The common order type on $B(1)$ - sublattices yields an ordering on $\{0,1\}$. Say that $0 < 1$, the case $1 < 0$ can be handled analogously.
We claim that $(0,1) < (1,0)$ for every $B(2)$ - sublattice. Assume to the contrary that $(1,0) < (0,1)$ for every $B(2)$ - sublattice. Consider any $B(3)$ - sublattice. According to the assumption it follows that $(1,0,1) < (0,1,0) < (0,0,1)$. Thus by transitivity $(1,0,1) < (0,0,1)$, contradicting that each $B(1)$ - sublattice is of type '$0 < 1$'.
Finally let $\hat{x} = (x_0,\ldots,x_{m-1})$ and $\hat{y} = (y_0,\ldots,y_{m-1})$ be any two 0-1 sequences. Say $x_\nu = y_\nu$ for all $\nu < i$, $x_i = 0$ and $y_i = 1$. As each $B(1)$ - sublattice is of type $0 < 1$ it follows that $\hat{x} \leq (x_0,\ldots,x_{i-1},0,1,\ldots,1)$ and $(y_0,\ldots,y_{i-1}),1,0,\ldots,0) \leq \hat{y}$. But then $\hat{x} < \hat{y}$ from the above considerations,

showing that $\leq = \leq^*$.

Again the minimality of Ω is obvious, in fact Ω is uniquely determined. □

(2.3) Points in parameter-sets over three-element alphabets

Parameter-sets have been introduced by Graham and Rothschild [2] as a tool for proving partition theorems. In a sense they may be viewed as a generalization of Boolean algebras to larger alphabets than just $\{0,1\}$. Let A be a finite alphabet, for our purposes if suffices to let $A = \{0,1,2\}$.

Let $[A]$ be a category which has as objects A^k , i.e. A-sequences $(x_0,\dots,x_{k-1})^T$ of length k , where k is a nonnegative integer. For $k \leq n$ let the morphisms $f \in [A]\binom{n}{k}$ correspond bijectively to k-parameter subsets of A^n , where a k-parameter subset of A^n is given by an A-sequence $\hat{x} = (x_0,\dots,x_{n-1})^T$ and by k mutually disjoint and nonempty sets $I_1,\dots,I_k$ which are subsets of $n = \{0,\dots,n-1\}$ such that $x_\nu = 0$ for every $\nu \in I_1 \cup\dots\cup I_k$. The k-parameter subset then consists of all A-sequences $y = (y_0,\dots,y_{n-1})^T \in A^n$ with $y_\nu = x_\nu$ for all $\nu \notin I_1 \cup\dots\cup I_k$ and $y_\nu = y_\mu$ for all $\nu,\mu \in I_i$ for some $i = 1,\dots,k$.
For $A = 2 = \{0,1\}$ the categories $[A]$ and $\mathcal{B}$ are isomorphic.
For $A = 3 = \{0,1,2\}$ a k-parameter subset in A^n can be interpreted as a k-dimensional affine subspace of $(GF(3))^n$, but generally not vice versa. Surprisingly the result here is somewhat different from the previous ones:

Theorem 3 Let $\leq \in ORD(A)$, say $a_0 < a_1 < a_2$. Consider the three orders $\leq^*$, $\leq^{**}$ and $\leq^{***}$ on A^m which are defined in the following way:

(1) $\leq^*$ is the lexicographic order.

(2) $(x_0,\dots,x_{m-1})^T <^{**} (y_0,\dots,y_{m-1})^T$ iff

   a) there exists an $i < m$ such that $x_\nu \in \{a_0,a_1\}$ iff $y_\nu \in \{a_0,a_1\}$ for every $\nu < i$, $x_i \in \{a_0,a_1\}$ and $y_i = a_2$ or

   b) $x_\nu = a_2$ iff $y_\nu = a_2$ for every $\nu < m$ and there exists an $i < m$ such that $x_\nu = y_\nu$ for every $\nu < i$ and $x_i < y_i$.

(3) $(x_0,\dots,x_{m-1})^T \leq^{***} (y_0,\dots,y_{m-1})^T$ iff

a) there exists an $i < m$ such that $x_\nu = a_0$ iff $y_\nu = a_0$ for every $\nu < i$, $x_i = a_0$ and $y_i \in \{a_1, a_2\}$ or

b) $x_\nu = a_0$ iff $y_\nu = a_0$ for every $\nu < m$ and there exists an $i < m$ such that $x_\nu = y_\nu$ for every $\nu < i$ and $x_i < y_i$.

Then $\Omega = \{\leq^*, \leq^{**}, \leq^{***} \mid \leq \in \mathrm{ORD}(A)\}$ is the uniquely determined set of canonical orders for $[A]\binom{m}{0}$.

We have good hope that analogous characterizations can be found also for larger alphabets. Proofs and details will appear somewhere else.

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