CHARACTERIZATION OF THE EXTREMAL RAYS OF THE CONES OF POSITIVE ELEMENTS IN TENSOR ALGEBRAS

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In this paper we give an explicit characterization of the extremal rays of the cones of positive elements in tensor algebras. This is an answer to a problem formulated in /3/, /1/.
In section 1 we give definitions and some basic properties of the cones of positive elements in tensor algebras. The fact that the cones of positive elements are generated by its extremal rays follows. In section 2 the extremal rays of the cones of positive elements are explicitly characterized. First we discuss the special case of the algebra of polynomials in one real variable and then the general case of tensor algebras. In section 3 we illustrate the results by some examples.

1. Tensor algebras and their cones of positive elements

Let $E[\tau]$ be a locally convex space with a continuous involution "*". Then the tensor algebra $E_{\otimes}$ over E is defined by

$$E_{\otimes} = \mathbf{C} \oplus E \oplus (E \otimes E) \oplus (E \otimes E \otimes E) \oplus \ldots ,$$

$\mathbf{C}$ is the field of complex numbers, i.e. the elements of $E_{\otimes}$ are of the type

$$f=(f_o,f_1,\ldots,f_N,0,0,\ldots)$$

with $f_i \in E^{\otimes i} = E \otimes E \otimes \ldots \otimes E$ (i times), $E^{\otimes o} = \mathbf{C}$.
$E_{\otimes}$ becomes a *-algebra with unity by the following componentwise defined algebraic operations:

$$(f+g)_n = f_n + g_n,$$

$$(fg)_n = \sum_{i+j=n} f_i \otimes g_j,$$

$$(f^*)_n = f^{(n)*} \otimes f^{(n-1)*} \otimes \ldots \otimes f^{(1)*} \quad \text{for}$$

for $f_n=f^{(1)}\otimes\ldots\otimes f^{(n)}$ and their continuous extension to $E^{\otimes n}$, $n=0,1,\ldots,$
$f_i,\ g_i\in E^{\otimes i}$, $i=0,1,2,\ldots$.

The element $\mathbf{1}=(1,0,0,\ldots)$ is the unity.
In the following the subindex n of $f_n^{(i)}$ denotes $f_n^{(i)}\in E^{\otimes n}$ and the index i of $f_n^{(i)}$ is a numeration index. Grad(f) denotes the highest index N with $f_N\neq 0$.
The cone of positive elements of $E_\otimes$ is defined by

$$E_\otimes^+=\left\{\sum_{i=1}^{M} f^{(i)}*f^{(i)};\ f^{(i)}\in E_\otimes,\ M\in\mathbf{N}\right\}.$$

$E_\otimes^+$ induces a semiordering on the hermitean part $E_\otimes^h=\{f\in E_\otimes;\ f=f^*\}$ of $E_\otimes$. This semiordering is given by $a\leq b$ iff $b-a\in E_\otimes^+$, $a,\ b\in E_\otimes^h$.

$$[a,b]=\{x\in E_\otimes^h;\ a\leq x\leq b\}=(a+E_\otimes^+)\cap(b-E_\otimes^+)$$

is the order intervall between a and b, see e.g. /15/. Further

$$\{\lambda k;\ k\in E_\otimes^+,\lambda\geq 0\}$$

is an extremal ray of $E_\otimes^+$ if $x\in[0,k]$ implies $x=\mu k,\ 0<\mu\leq 1$.
This is equivalent to the condition that $k=k^{(1)}+k^{(2)}$, $k^{(1)},\ k^{(2)}\in E_\otimes^+$, implies $k=\mu_1 k^{(1)}=\mu_2 k^{(2)}$, $\mu_1,\mu_2>0$.
Let us regard the projective tensor product topology τ_n on $E^{\otimes n}$ and the direct tensor product topology $\tau_\otimes$ of $\bigoplus_{n=0}^{\infty} E^{\otimes n}[\tau_n]$ on $E_\otimes$. Some properties of the topologies on tensor algebras are investigated in /9/, /11/, /21/.
The physical motivation for the investigation of tensor algebras comes from quantum field theory and is due to /2/, /18/.

First let us prove the following key lemma.

Lemma 1.1:
Let $f_n^{(j)}\in E^{\otimes n}$ and $\sum_{j=1}^{M} f_n^{(j)}*f_n^{(j)}=0$ for a fixed index n.
Then $f_n^{(1)}=f_n^{(2)}=\ldots=f_n^{(M)}=0$ follows.

Proof:
Let $f_n^{(1')}\neq 0$ and T_n be a hermitean linear functional on $E^{\otimes n}$ with $T_n(f_n^{(1')})\neq 0$, $1'\in\{1,2,\ldots,M\}$. Such functionals exist because of the theorem of Hahn-Banach and the isomorphism α between the hermitean linear functionals W on $E_\otimes$ and the real linear functionals (αW) on $E_\otimes^h$ given by $W(g)=\frac{1}{2}(\alpha W(g+g^*)+i\,\alpha W(g-g^*))$. Then

$$T_n\otimes T_n\left(\sum_{j=1}^{M} f_n^{(j)}*f_n^{(j)}\right)=\sum_{j=1}^{M}|T_n(f_n^{(j)})|^2\geq|T_n(f_n^{(1')})|^2>0.$$

But this is a contradiction to $\sum_{j=1}^{M} f_n^{(j)*} f_n^{(j)}=0$.

The central properties of $E_\otimes^+$ are sumed up in the following theorem. The notions are in the sense of /10/, /15/.

Theorem 1.2:

a) $E_\otimes^+$ is a proper and generating cone in $E_\otimes^h$.

b) $E_\otimes^h$ has no order unity with respect to the semiordering induced by $E_\otimes^+$. Further $E_\otimes^h$ does not have the Riesz decomposition property. Thus $E_\otimes^h$, $\leq$ is no vector lattice.

If $E[\tau]$ is a nuclear LF-space, then

c) $\bar{E}_\otimes^+ = \left\{ \sum_{i=1}^{\infty} f^{(i)*} f^{(i)};\ f^{(i)} \in E_\otimes,\text{ the sum is } \tau_\otimes\text{-convergent} \right\}$

($\bar{\ }$ denotes the topological closure with respect to $\tau_\otimes$ in the completed tensor algebra $\tilde{E}_\otimes$).

d) $\bar{E}_\otimes^+$ is a proper and generating cone in $\tilde{E}_\otimes^h$.

e) $E_\otimes^+$, $\bar{E}_\otimes^+$ have no topological interior points.

f) $E_\otimes^+$ is not locally compact.

g) The order intervalls with respect to $\bar{E}_\otimes^+$ are compact sets,/1/.

h) There are locally convex topologies η on $E_\otimes$ with the property that $E_\otimes^+$ is η-normal and $\eta\restriction_{E^{\otimes n}} = \tau_n$.

Proof:

a) $k, -k \in E_\otimes^+$ implies $k=0$ by lemma 1.1.
Because of $f= \frac{1}{4}((1+f)^*(1+f)-(1-f)^*(1-f))$ for $f=f^*$ we have $E_\otimes^h=E_\otimes^+-E_\otimes^+$.

b) Let $k \in E_\otimes^+$ be an order unity with Grad(k)=2N. Further let $f \in [0,k]$, Grad(f)=2M. But $M>N$ implies $-f_{2M} \in E_\otimes^+$ which is a contradiction to a). Thus $M \leq N$ and k is no order unity.
Now let $x=2g_1 \otimes g_1$, $y=2g_2 \otimes g_2$ with $g_1=g_1^* \in E$, $g_2=g_2^* \in E^{\otimes 2}$. Then $h+(0,g_1,g_2,0,\dots)(0,g_1,g_2,0,\dots)=x+y$ for $h=(0,g_1,-g_2,0,\dots)(0,g_1,-g_2,0,\dots)$. Thus $h \in [0,x+y]$. $f \in [0,x]$ and $g \in [0,y]$ imply $f=(0,0,f_2,0,\dots)$ and $g=(0,0,0,0,g_4,0,\dots)$. Thus $h \notin [0,x] + [0,y]$ and the Riesz decomposition property is not valid.

c) The proof for $E=\mathcal{S}$ (Schwartz space) is contained in /4/, /7/. Afterwards other proofs were given in /1/, /16/.

d) is a consequence of a) and c).

e) In every $\tau_\otimes$-neighbourhood we find elements $g \in E_\otimes$ with the property that Grad(g) is an odd number. But a consequence of $k \in E_\otimes^+$ is that Grad(k) is even. Thus every $\tau_{\otimes}$-neighbourhood has elements not contained in $E_{\otimes}^{+}$, /20/. Because of c) the proof is valid for $E_{\otimes}^{+}$, too.

f) is a consequence of e).

g) Because of h) and /15,V.3.1/ the order intervalls are bounded. The statement follows by the nuclearity of $E_{\otimes}[\tau_{\otimes}]$ and /14, 4.4.7, 5.2.2/.

h) was proved in /8/, /9/, /11/, /21/.

Remarks 1.3:

a) Because of theorem 1.2.f) the theorem of Klee (/10/) on the representation of a cone by its extremal rays is not applicable to the cone of positive elements in tensor algebras.

b) $\tilde{E}_{\otimes}[\tau_{\otimes}]$ is complete by /15, II.6.2/ and dualnuclear by /14,4.3.2/. Thus $E_{\otimes}^{+}$ is the closed convex hull of its extremal rays by a theorem of E.G.F.Thomas (/7/), cf. /1/, /22/.

c) Because of theorem 1.2.c) every extremal ray of $E_{\otimes}^{+}$ is an extremal ray of $\bar{E}_{\otimes}^{+}$ and vice versa.

2. The extremal rays of the cones of positive elements

a) The algebra of polynomials $\mathbf{C}_{\otimes}$

Let us start with the simplest case of a tensor algebra. This is the tensor algebra over the field of complex numbers $\mathbf{C}$. $\mathbf{C}_{\otimes}$ is isomorphic to the algebra of polynomials in one real variable t and with complex coefficients. The following isomorphism holds:

$$\mathbf{C}_{\otimes} \ni p=(p_0,p_1,\ldots,p_N,0,\ldots) \longleftrightarrow \hat{p}(t)=p_0+p_1t+p_2t^2+\ldots+p_Nt^N,$$

$$p^* \longleftrightarrow \overline{\hat{p}(t)},$$

($\overline{\hat{p}(t)}$ means the complex conjugated value).

Theorem 2.1:

a) $p \in \mathbf{C}_{\otimes}^{+}$ iff $\hat{p}(t) \geqslant 0$ for all $t \in \mathbf{R}$.

b) $\mathbf{C}_{\otimes}^{+} = \bar{\mathbf{C}}_{\otimes}^{+}$.

c) The element $p=c^*c$, $c \in \mathbf{C}_{\otimes}$, is an extremal ray iff $\hat{c}(t)$ has only real roots.

Proof:

a),b) were proved in /12/.

c) We assume that $\hat{c}(t)$ has the decomposition

$$\hat{c}(t) = \prod_{j=1}^{r} (t-(a_j+ib_j)) \prod_{j=r+1}^{n} (t-a_j),\ a_j,\ b_j \in \mathbf{R},$$

i is the imaginary unity. Then

$$c*c \longleftrightarrow \prod_{j=1}^{r} ((t-a_j)^2+b_j^2) \prod_{j=r+1}^{n} (t-a_j)^2 =$$

$$= \prod_{j=1}^{n}(t-a_j)^2 + \sum_{s=1}^{r} b_s^2 \prod_{\substack{j=1\\ s\neq j}}^{n} (t-a_j)^2 + \sum_{\substack{s,t=1\\ s<t}}^{r} b_s^2 b_t^2 \prod_{\substack{j=1\\ s\neq j\\ t\neq j}}^{n} (t-a_j)^2 + \dots + \qquad (1)$$

$$\dots + b_1^2 \dots b_r^2 .$$

If $\hat{c}(t)$ has complex roots, i.e. $r \geq 2$, then

$$c*c = x^{(1)}*x^{(1)}+x^{(2)}*x^{(2)}+\dots+x^{(m)}*x^{(m)},\ m \geq 2,\ x^{(1)},\dots,x^{(m)} \in \mathbf{C}_{\otimes}.$$

Thus $c*c$ is no extremal ray.

Now let $\hat{c}(t)$ have only real roots, i.e. $\widehat{c*c}(t) = \prod_{j=1}^{n} (t-a_j)^2$, $a_j \in \mathbf{R}$, and we assume that $c*c$ is no extremal ray. That means that

$$c*c = x^{(1)}*x^{(1)}+x^{(2)}*x^{(2)},\ c \neq \mu_j x^{(j)},\ \mu_j > 0,\ j=1,2.$$

On the other side $\widehat{x^{(j)}*x^{(j)}}(t) = \prod_{r=1}^{n} (t-a_r)^2\, \hat{p}^{(j)}(t)$, $\hat{p}^{(j)}(t) \geq 0$, for all $t \in \mathbf{R}$, $j=1,2$, and $\hat{p}^{(1)}(t)+\hat{p}^{(2)}(t)=1$ has to be fulfilled. This is a contradiction to our assumption.

Remarks 2.2:

a) Because the elements on the right hand side of equation (1) are extremal rays equation (1) gives a decomposition of an arbitrary $k \in \mathbf{C}_{\otimes}^{+}$ into extremal ones. This decomposition contains only finite many terms. Thus this is a sharper statement than that which followed by the theorem of E.G.F.Thomas, cf. remark 1.3.b).

b) Let $f=(f_0,f_1,\dots,f_N,0,\dots) \in E_{\otimes}$. f is called to be generated by one element f_{i_0} if

$$f_j = \begin{cases} \lambda_j\, f_{i_0} \otimes \dots \otimes f_{i_0} & \text{for } j=mi_0 \\ 0 & \text{otherwise} \end{cases}$$

$\lambda_j \in \mathbf{C}$, $j,m=1,2,\dots$.

Let us regard the *-homomorphism β from $E_{\otimes}$ into $\mathbf{C}_{\otimes}$ given by

$$\beta(f) = (f_0,c_1,c_2,\dots,c_N,0,\dots.)$$

with

$$c_j = \begin{cases} \lambda_j & \text{if } f_j = \lambda_j f_{i_0} \otimes \dots\dots \otimes f_{i_0} \\ 0 & \text{otherwise} \end{cases} .$$

If $E[\tau]$ is a nuclear LF-space then $\beta(E_{\otimes}^{+})\subset \mathbf{C}_{\otimes}^{+}$ and $\beta(\bar{E}_{\otimes}^{+})\subset \mathbf{C}_{\otimes}^{+}$.

c) A consequence of theorem 2.1 is the following assertion:

Let f be generated by one element f_{i_o} . Then f*f is an extremal ray of $E_{\otimes}^{+}$ iff $\widehat{\beta(f)}$ has only real roots.

d) $p\leq q$ iff $\hat{p}(t)\leq\hat{q}(t)$ for all $t\in\mathbf{R}$. Thus the order intervall $[0,p]$ is given by

$$\{q\in \mathbf{C}_{\otimes};\ 0\leq\hat{q}(t)\leq\hat{p}(t)\ \text{ for all }\ t\in\mathbf{R}\}.$$

Consequently $q\in[0,p]$ implies $\mathrm{Grad}(q)\leq\mathrm{Grad}(p)$. The following two examples illustrate the theorem 2.1:

i) Let $\hat{p}(t)=t^4-2t^2+1$. Then $\hat{p}_1(t)\in[0,\hat{p}]$ iff $\hat{p}_1(t)=c(t^4-2t^2+1)$, $0\leq c\leq 1$, see figure 1.

ii) Let $\hat{q}(t)=t^2+1$, $\hat{q}_1(t)=t^2$, $\hat{q}_2=1$, $\hat{q}_3(t)=\frac{1}{4}t^2-\frac{1}{2}t+\frac{3}{4}$. It is $q_i\in[0,q]$, i=1,2,3, and $q_1+q_2=q$, see figure 2. Further $q=c*c$ with $c=(i,1,0,\dots)$.

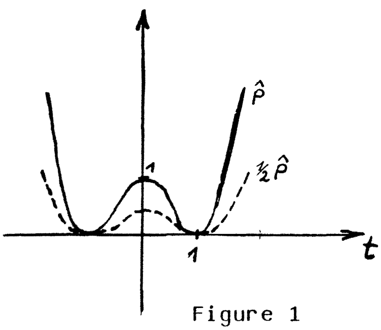

Figure 1

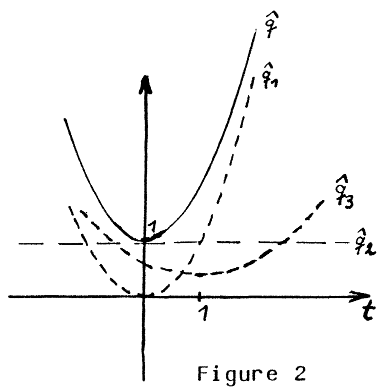

Figure 2

b) Arbitrary tensor algebras

In this section we prove an explicit characterization of the extremal rays of the cones of positive elements in tensor algebras over arbitrary vector spaces. Let us start with the following lemma.

Lemma 2.3:

Let $f_n,\ g_n^{(i)}\in E^{\otimes n}$ and $f_n^*\otimes f_n=\sum_{i=1}^{M} g_n^{(i)*}\otimes g_n^{(i)}$, n=1,2,... .

Then $g_n^{(i)}=\lambda_n^{(i)}f_n$, $\lambda_n^{(i)}\in\mathbf{C}$, i=1,2,...,M, holds.

Proof:

Let $F_n=\{cf_n;\ c\in\mathbf{C}\}\subset E^{\otimes n}$, n=1,2,... . Let us regard a direct decomposition $E^{\otimes n}=F_n\oplus G_n$ and the corrosponding projectors P, Q.

$g_n^{(i)} = Pg_n^{(i)} \oplus Qg_n^{(i)}$ and thus

$$\sum_{i=1}^{M} g_n^{(i)*} \otimes g_n^{(i)} = (\sum_{i=1}^{M} (Pg_n^{(i)})^* \otimes Pg_n^{(i)}) \oplus (\sum_{i=1}^{M} (Pg_n^{(i)})^* \otimes Qg_n^{(i)}$$

$$+ (Qg_n^{(i)})^* \otimes Pg_n^{(i)}) \oplus (\sum_{i=1}^{M} (Qg_n^{(i)})^* \otimes Qg_n^{(i)}).$$

$\sum_{i=1}^{M} g_n^{(i)*} \otimes g_n^{(i)} \in F_n^* \otimes F_n$ holds by the assumption. Hence

$$\sum_{i=1}^{M} (Qg_n^{(i)})^* \otimes Qg_n^{(i)} = 0.$$

This implies $Qg_n^{(i)} = 0$, $i=1,2,\dots,N$, by lemma 1.1.

A consequence of lemma 2.3 is the following corollary.

Corollary 2.4:

The elements $f_n^* \otimes f_n$, $n=1,2,\dots$ are extremal rays of $E_\otimes^+$.

Lemma 2.5:

Let $f=(f_0,f_1,\dots,f_N,0,\dots)$, $g^{(i)}=(g_0^{(i)},g_1^{(i)},\dots,g_{N_i}^{(i)},0,\dots)\in E_\otimes$ and $f^*f=\sum_{i=1}^{M} g^{(i)*}g^{(i)}$. Then $g_j^{(i)} = \lambda_j^{(i)} f^{(j)}$, $\lambda_j^{(i)} \in \mathbf{C}$, $j=0,1,\dots,N$, $i=1,2,\dots,M$.

Proof:

$\mathrm{Grad}(g^{(i)}) = N_i \leq N$ follows by lemma 1.1 and $g_N^{(i)} = \lambda_N^{(i)} f_N$, $i=1,2,\dots,M$, follows by lemma 2.3. Now let $g_k^{(i)} = \lambda_k^{(i)} f_k$ for $k=r+1,r+2,\dots,N$, $r \in \mathbf{N}$. We put $F_j = \{cf_j;\ c \in \mathbf{C}\}$, $j=1,2,\dots$. Let $g_r^{(i')} \notin F_r$ for an $i' \in \{1,2,\dots,M\}$. On $E^{\otimes r}$ we define a linear hermitean functional T_r with $T_r(F_r)=0$,

$$T_r \otimes T_r(\sum_{\substack{l+k=2r \\ l<k}} (f_l^* \otimes (f_k - \sum_{i=1}^{M} \bar{\lambda}_l^{(i)} g_k^{(i)}) + (f_k^* - \sum_{i=1}^{M} \lambda_l^{(i)} g_k^{(i)*}) \otimes f_l)) = 0$$

and $T_r(Qg_r^{(i')}) \neq 0$ with a decomposition $E^{\otimes r} = F_r \oplus G_r$ and the corrosponding projectors P, Q. Then

$$T_r \otimes T_r(\sum_{l+k=2r} f^*_l \otimes f_k - \sum_{i=1}^{M} \sum_{l+k=2r} g_l^{(i)*} \otimes g_k^{(i)}) =$$

$$-T_r \otimes T_r(\sum_{i=1}^{M} (Qg_r^{(i)})^* \otimes (Qg_r^{(i)}) \leq -T_r \otimes T_r((Qg_r^{(i')})^* \otimes (Qg_r^{(i')})) =$$

$$-|T_r(Qg_r^{(i')})|^2 < 0 .$$

But this is a contradiction to our assumption and thus $g_r^{(i)} = \lambda_r^{(i)} f_r$, $i=1,2,\ldots,M$. The assertion of the lemma follows by finite many steps.

In the following let $f=(f_0,f_1,\ldots,f_N,0,\ldots)\in E_\otimes$, $Grad(f)=N$,

$r'=\max(r-N,0)$,

$$M_r=\{(r',r-r'),(r'+1,r-r'-1),\ldots,([r/2],r-[r/2])\},$$

$$L_r=\{(u,v)\in M_r;\ Re(f_u^*\otimes f_v)=0\},$$

where $[r/2]$ denotes the greatest integer which is less or equal to $r/2$ and $Re(f_u^*\otimes f_v)=\frac{1}{2}(f_u^*\otimes f_v + f_v^*\otimes f_u)$. Further let

$$\Big\{\sum_{(u,v)\in M_r} \varrho_{u,v}^i\, Re(f_u^* \times f_v)=0;\ \varrho_{u,v}^i\in \mathbf{R},\ i=1,2,\ldots,l_r\Big\}$$

be a maximal system of linear independent nontrivial relations between the elements $Re(f_u^*\otimes f_v)$, $(u,v)\in M_r$, $\varrho_{u,v}^i\in\mathbf{R}$.

We put $\hat{l}=l_2+l_3+\ldots+l_{2N-2}$. Now let us regard the $(N+1,N+1)$-matrix $\mathbf{S}(t_1,\ldots,t_{\hat{l}})$ with the elements

$$s_{p,q}=\begin{cases}\frac{1}{2}(\varrho_{p,q}^1 t_{l_{p+q-1}+1}+\ldots+\varrho_{p,q}^{l_{p+q}}\, t_{l_{p+q-1}+l_{p+q}}) & \text{for } p<q\\ s_{q,p} & \text{for } q<p\\ \varrho_{p,p}^1\, t_{l_{2p-1}+1}+\ldots+\varrho_{p,p}^{l_{2p}}\, t_{l_{2p-1}+l_{2p}} & \text{for } p=q\end{cases}$$

$p,q=0,1,\ldots,N$, depending of the real parameters $t_1,t_2,\ldots,t_{\hat{l}}$.

Further let $\mathbf{A}$ be a $(N+1,N+1)$-matrix with the elements

$$a_{p,q}=\begin{cases}1 & \text{if } (p,q)\in M_r\setminus L_r\\ 0 & \text{if } (p,q)\in L_r\,,\qquad p,q=0,1,\ldots,N.\end{cases}$$

Then

$$f^*f=\sum_{p,q=0}^{N}(a_{p,q}+s_{p,q}(t_1,\ldots,t_{\hat{l}}))\ Re(f_p^*\otimes f_q) \tag{2}$$

holds for arbitrary $t_i\in\mathbf{R}$, $i=1,2,\ldots,\hat{l}$.

Theorem 2.6:

$\{\mu f^*f;\ \mu\geq 0\}$ is no extremal ray iff there are $t_i^o\in\mathbf{R}$, $i=1,2,\ldots,\hat{l}$, such that the matrix $\mathbf{A}+\mathbf{S}(t_1^o,\ldots,t_{\hat{l}}^o)$ is positive semidefinit and $rk(\mathbf{A}+\mathbf{S}(t_1^o,\ldots,t_{\hat{l}}^o))\geq 2$.

($rk(.)$ denotes the rank of a matrix .).

Proof:

a) Let $\mathbf{S}^o=\mathbf{S}(t_1^o,\ldots,t_{\hat{l}}^o)$ and $\mathbf{U}$ be a real symmetric matrix which diagonalizes the matrix $\mathbf{A}+\mathbf{S}^o$, i.e.

$$\mathbf{U}^*(\mathbf{A}+\mathbf{S}^o)\mathbf{U}=\begin{pmatrix}\mu_o & & & O\\ & \mu_1 & & \\ & & \ddots & \\ O & & & \mu_N\end{pmatrix}$$

with $\mu_o,\ldots,\mu_N \geqslant 0$, $\mu_{i_1},\ldots,\mu_{i_s} > 0$, $i_1,\ldots,i_s \in \{0,1,\ldots,N\}$, $s=\mathrm{rk}(\mathbf{A}+\mathbf{S}^o) \geqslant 2$. We put

$$\mathbf{c}^{(i)}=(c_o^{(i)},\ldots,c_N^{(i)})=(0,\ldots,0,\mu_i,0,\ldots,0)\ \mathbf{U}\ .$$

Then we get

$$a_{p,q}+s^o_{p,q} = \sum_{i=1}^{N} c_p^{(i)}\ c_q^{(i)}$$

and the decomposition $f*f=g^{(1)}*g^{(1)}+\ldots+g^{(N)}*g^{(N)}$ with $g^{(i_1)}\neq 0$, $\ldots,g^{(i_s)}\neq 0$, $s\geqslant 2$, for $g^{(i)}=(c_o^{(i)}f_o,c_1^{(i)}f_1,\ldots,c_N^{(i)}f_N,0,\ldots)$ by equation (2).

b) Lemma 2.5 implies that every decomposition of f*f is of the type (2) because otherwise there would be some new independent relations between the elements $\mathrm{Re}(f_u^* \otimes f_v)$. Thus f*f can not be decomposed if there do not exist parameters $t_i^o \in \mathbf{R}$, $i=1,2,\ldots,\hat{1}$, with $\mathbf{A}+\mathbf{S}^o$ is positive semidefinit and $\mathrm{rk}(\mathbf{A}+\mathbf{S}^o) \geqslant 2$.

Remark 2.7:
We constructed a decomposition of f*f in part a) of the proof. But the items of these decompositions are no extremal ones in general. We investigate this problem in a subsequent paper.

3. Examples

We demonstrate the main theorem by some examples.

a)
Let $f=(1,f_1,0,\ldots)\in E_\otimes$ with $f_1\neq 0$ and $\mathrm{Re}(f_1)=0$. Thus $\mathbf{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\mathbf{S}(t)=0$. f*f is decomposable because $\mathbf{A}$ is positive definit and $\mathrm{rk}(\mathbf{A})=2$. This decomposition is given by $f*f=g^{(1)}*g^{(1)}+g^{(2)}*g^{(2)}$ with $g^{(1)}=(1,0,\ldots)$, $g^{(2)}=(0,f_1,0,\ldots)$.

b)
Now let $f=(1,f_1,f_2,0,\ldots)$ with $f_1^*\otimes f_1=4\mathrm{Re}(f_2)\neq 0$, $\mathrm{Re}(f_1)\neq 0$, $\mathrm{Re}(f_1^*\otimes f_2)\neq 0$, $f_2^*\otimes f_2\neq 0$. Then

$$\mathbf{A}+\mathbf{S}(t)=\begin{pmatrix}1 & 1 & 1-2t\\ 1 & 1+t & 1\\ 1-2t & 1 & 1\end{pmatrix}$$

$\mathbf{A}+\mathbf{S}(t)$ is positive semidefinit iff $t\geqslant 0$. $\mathrm{rk}(\mathbf{A}+\mathbf{S}(t))=3$ for $t>0$ and $\mathrm{rk}(\mathbf{A}+\mathbf{S}(0))=\mathrm{rk}(\mathbf{A})=1$.

Let us construct the decomposition for t=1/2. The eigen values of $\mathbf{A}+\mathbf{S}(1/2)$ are $\mu_1=1$, $\mu_2=\frac{1}{4}(5-\sqrt{33})$, $\mu_3=\frac{1}{4}(5+\sqrt{33})$ and the normed eigen vectors are $\mathbf{c}^{(1)}=(1/\sqrt{2},0,-1/\sqrt{2})$, $\mathbf{c}^{(2)}=2\sqrt{2}\,\sqrt{33-\sqrt{33}}^{\,-1}(1,\frac{1}{4}(1-\sqrt{33}),1)$, $\mathbf{c}^{(3)}=2\sqrt{2}\,\sqrt{33+\sqrt{33}}^{\,-1}(1,\frac{1}{4}(1+\sqrt{33}),1)$. Hence the decomposition

$$f*f=\sum_{i=1}^{3} g^{(i)}*g^{(i)} \quad \text{with} \quad g^{(1)}=(1/\sqrt{2},0,(-1/\sqrt{2})f_2,0,0,\ldots),$$

$$g^{(2)}=2\sqrt{2}\sqrt{33-\sqrt{33}}^{\,-1}(1,\tfrac{1}{4}(1-\sqrt{33})f_1,f_2,0,\ldots)$$

$$g^{(3)}=2\sqrt{2}\sqrt{33-\sqrt{33}}^{\,-1}(1,\tfrac{1}{4}(1+\sqrt{33})f_1,f_2,0,\ldots.)$$

holds.

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REFERENCES

/1/ ALCANTARA J. "C*-Algebras techniques in tensor algebras", preprint, The Open University (1980).

/2/ BORCHERS H.J. "On the structure of the algebra of field operators", Nuovo Cimento 24 (1962), 214.

/3/ BORCHERS H.J. "Algebraic aspects of Wightman field theory", in: Sen&Weil (Eds.), Statistical mechanics and field theory, New York: Halsted Press 1972.

/4/ BORCHERS H.J. "On the algebra of test function", prebublications de la RCP 25/15 (1973).

/5/ BRAUER F. "Über die Struktur des positiven Kegels in der Feld-algebra", Diplomarbeit, Göttingen 1973.

/6/ HOFMANN G. "Die Testfunktionenalgebra $\mathcal{S}_{\otimes}$ und ihre Anwendung in der axiomatischen Quantenfeldtheorie", Dissertation, Leipzig 1975.

/7/ HOFMANN G. "The closure of cones in the algebra of test functions", Rep.Math.Phys. 13 (1978), 187.

/8/ HOFMANN G. "Ein hinreichendes Kriterium für die Normalität von Kegeln in Tensoralgebren und einige Anwendungen", Wiss.Z. Karl-Marx-Univ. Leipzig, Math.-Naturwiss.R. 27 (1978), 261.

/9/ HOFMANN G. "Topologies on the algebra of test functions", Dubna-Preprint, JINR E2-10763 (1977).

/10/ KÖTHE G. "Topologische lineare Räume", Springer Verlag 1960.

/11/ LASSNER G. " O* -topologies on the test functions algebra", Publ.Depart.Math.Lyon, 12.1 25 (1975).

/12/ LASSNER G. "Über die Realisierbarkeit topologischer Tensoralgebren", Math.Nachr. 62 (1974), 89.

/13/ LASSNER G.; UHLMANN A "On positive functionals on the algebra of test functions", Comm.Math.Phys. 7 (1968), 152.

/14/ PIETSCH A. "Nukleare lokalkonvexe Räume", Akademieverlag Berlin 1969.

/15/ SCHÄFER H.H. "Topological vector spaces", New York 1968.

/16/ SCHMÜDGEN K. "Revue Roum.Math. Peres et Appl. (to appear).

/17/ THOMAS E.G.F. "Repreśentations intégrale danes les cônes convexes", C.R.Acad. Sc. Paris, t.286 (1978), Serie A, 515.

/18/ UHLMANN A. "Über die Definition der Quantenfelder nach Wightmann und Haag", Wiss.Z.Karl-Marx-Univ.Leipzig, Math.-Naturwiss.R., 11 (1962), 213.

/19/ UHLMANN A. ""Über die Algebra der Feldoperatoren", preprint Leipzig 1963.

/20/ WYSS W. "The field algebra and its positive linear functionals", Comm.Math.Phys. , 26 (1972), 223.

/21/ YNGVASON J. "On the algebra of test functioons for field operators", Comm.Math.Phys., 34 (1973), 315.

/22/ YNGVASON J. "Über den algebraischen Formalismus in der Wighmanschen Quantenfeldtheorie", Habilitationsschrift, Göttingen 1978.

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