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# TWO QUESTIONS CONCERNING VECTOR-VALUED HOLOMORPHIC FUNCTIONS

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Let $\Omega \subset \mathbb{C}$ be an open and connected subset and $E$ a complex Hausdorff locally convex space which is locally complete, i.e.: every bounded set is contained in a bounded Banach-disc. $H(\Omega,E)$ denotes the set of all holomorphic (= weakly holomorphic) functions $\Omega \to E$. For a Banach-disc $B$ and its Minkowski-gauge-functional $m_B$ write $[\![B]\!] := (\text{span } B, m_B)$ for the Banach-space which is spanned up by $B$.

(P1) GLOBAL FACTORIZATION: *Characterize those pairs* $(\Omega,E)$ *such that for every* $f \in H(\Omega,E)$ *there is a bounded Banach-disc* $B$ *in* $E$ *such that* $f : \Omega \to [\![B]\!] \hookrightarrow E$ *holomorphically.*

(P2) INTERPOLATION: *Given a sequence* $(z_n)$ *in* $\Omega$ *which is discrete in* $\Omega$ *and a sequence* $(x_n)$ *in* $E$. *Under which circumstances does* $f \in H(\Omega,E)$ *exist with* $f(z_n) = x_n$ *for all* $n \in \mathbb{N}$ *?*

It is well-known that every $f \in H(\Omega,E)$ factors *locally* through a compact Banach-disc. So it is easy to see

(1) *If* $E'_{co}$ *has the countable neighbourhood property* (i.e. for every sequence $(U_n)$ of neighbourhood of zero there are $\lambda_n > 0$ such that $\bigcap_n \lambda_n U_n$ is a neighbourhood of zero) *then* $H(\Omega,E)$ *admits global factorization for all* $\Omega$. *The Banach-disc for the factorization can be chosen even to be compact.*

This is true e.g. for Fréchet-spaces $E$. As a consequence

(2) *If* $f \in H(\Omega,E)$ *factors globally it factors globally through a Banach-space* $[\![B]\!]$ *where* $B$ *is a compact Banach-disc.*

If $E = \hat{H}(\{0\})$ , the nuclear (LS)-space of germs of holomorphic functions in $\{0\} \subset \mathbb{C}$ , then the holomorphic function $f : \mathbb{C} \setminus \{0\} \to \hat{H}(\{0\})$ defined by

$$f(z) = \frac{1}{z-\cdot}$$

(the "moving-pole function") admits no global factorization.

(3) *If* $A \subset \Omega$ *is a set of uniqueness for holomorphic functions in* $H(\Omega)$ *and* $F \subset E$ *is a closed subspace, then every* $f \in H(\Omega,E)$ *with* $f(A) \subset F$ *satisfies* $f(\Omega) \subset F$ .

This follows from a simple Hahn-Banach-argument. In particular, there is for every $f \in H(\Omega,E)$ a compact set $K \subset E$ (namely $f(\overline{K(z_o,\varepsilon)})$ ) such that $f(\Omega) \subset \overline{\text{span}\, K}$ . A Baire-argument (in $\Omega$ ), together with (3), implies

(4) *If* $E = \bigcup_{n=1}^{\infty} E_n$ *where* $E_n$ *are closed subspaces of* $E$ *, then there is for every* $f \in H(\Omega,E)$ *an* $n_o \in \mathbb{N}$ *such that* $f(\Omega) \subset E_{n_o}$ .

This answers the interpolation question to the negative in the case of strict inductive limits $E$ , e.g. there is no entire function $f$ with values in $\varphi := \underset{m\to}{\text{ind}}\, \mathbb{C}^m$ such that $f(n) = e_n$ (the n-th unit vector) for all $n \in \mathbb{N}$ . The same trick as for scalar-valued functions gives

(5) *If* $\frac{\partial}{\partial \bar{z}} : \mathcal{E}^\infty(\Omega,E) \to \mathcal{E}^\infty(\Omega,E)$ *is surjective, then there is always an interpolating function* $f$ *(as in* (P2)*).*

For Fréchet-spaces $E$ the $\frac{\partial}{\partial \bar{z}}$ -operator is always surjective: since $\mathcal{E}^\infty(\Omega) \tilde{\otimes}_\pi E = \mathcal{E}^\infty(\Omega,E)$ and the scalar $\frac{\partial}{\partial \bar{z}}$ -operator on $\mathcal{E}^\infty(\Omega)$ is onto this is a consequence of the fact that the $\pi$-tensorproduct of surjective operators between Fréchet-spaces is surjective. However, in general, $\frac{\partial}{\partial \bar{z}}$ is not onto, e.g. not on $\mathcal{E}^\infty(\mathbb{C},H(K(0,1)))$; but it is onto on $\mathcal{E}^\infty(\mathbb{C},H(\{0\}))$ (see [3], p. 23). In the class of (DF) -spaces $E$ (more general: of those spaces with a countable basis of bounded sets) D. Vogt [3] characterized the surjectivity of $\frac{\partial}{\partial \bar{z}}$ on $\mathcal{E}^\infty(\mathbb{C},E)$ in terms of the existence of a somehow dominating bounded set in $E$ .

R. M. Aron, J. Globevnik, and M. Schottenloher [1] investigated the *bounded* interpolation problem on $\Omega := K(0,1)$ : which are the sequences $(z_n)$ in $K(0,1)$ such that for every *bounded* sequence $(x_n)$ in $E$ there is a *bounded* $f \in H(K(0,1),E)$ which interpolates. Their result is: the sequences $(z_n)$ are the same for all Banach-spaces $E$ (different from zero), i.e. the same as in the scalar case.

## REFERENCES

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