Zsolt Tuza

τ - critical hypergraphs

In: Zdeněk Frolík (ed.): Abstracta. 9th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1981. pp. 181--184.

Persistent URL: http://dml.cz/dmlcz/701250

NINTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1981)

τ-critical hypergraphs

Zsolt Tuza

The vertex set and edge set of the hypergraph $\mathcal{H}$ are denoted by V and $\mathcal{E}$ respectively. /We suppose that $\mathcal{E} = \{E_1, \dots, E_m\}$ ./ $\mathcal{H}$ is r-uniform if $|E_i| = r$ for every i. The set T is a transversal of $\mathcal{H}$ if T intersects every E_i. The transversal number τ is the minimal size of a transversal of $\mathcal{H}$.

The hypergraph is called τ-critical if the deletion of any edge makes the transversal number decrease. It means that for every edge $E_i \in \mathcal{E}$ one can find a T_i with size $\tau-1$ that intersects every edge different from E_i. That is, we have a system of pairs of sets E_i and T_i such that $E_i \cap T_j = \emptyset$ iff $i=j$. For such systems Bollobás [2] proved

$$\sum_{i=1}^{m} \binom{|E_i| + |T_i|}{|E_i|}^{-1} \leq 1 \qquad /1/$$

/However, he proved this inequality in another non-symmetric form and he used the term of saturated graphs. Therefore his result remained almost unknown and was re-discovered by Katona and Tarján, cf. [5] ./

Though /1/ implies immediately that an r-uniform τ-critical hypergraph can have at most $\binom{r+\tau-1}{r}$ edges, this bound was thought to be unknown up to 1971 when Jaeger and Payan gave another proof on it /cf. [1, p.424]/. Later L.Lovász found a generalization on geometrical hypergraphs.

Erdős and Gallai [3] began studying τ-critical graphs. They proved that the size of the vertex set of a τ-critical graph is at most 2τ. Considering 3-uniform hypergraphs, a deep method of Szemerédi and Petruska [6] shows $|V| \leq 8\tau^2$. In the general r-uniform case /1/ would imply $|V| \leq c_r \tau^r$. However, the right order of magnitude is τ^{r-1}.

<u>Theorem</u> For r-uniform τ-critical hypergraphs

$$|V| \leq \tau^{r-1} + \tau\binom{\tau+r-2}{r-2}$$

Here we cketch the proof /further details can be found in [4] /.

If $\mathcal{T}$ is a collection of τ-element transversals T, we can examine $\tau_i = \min\left(|E_i'| : E_i' \subset E_i \text{ and } E_i' \text{ intersects every } T \in \mathcal{T} \right)$ we say $\mathcal{T}$ is good if $\tau_i \geq r-1$ for every i. Obviously, the set of all transversals of $\mathcal{H}$ is good.

Consider a minimal good collection i.e. the deletion of any T makes some τ_i decrease. In this situation for every $T_j \in \mathcal{T}$ we can find an E'_j with size r-2 such that $T_j \cap E'_k = \emptyset$ iff $j=k$. Therefore /1/ implies $|\bigcup T_j| \leq \tau \binom{\tau+r-2}{r-2}$. Since the set $T_o = \bigcup_i T_i$ intersects every edge of $\mathcal{H}$ in at most r-1 points, the set $A = V \setminus T_o$ is a strong stable set /i.e. it meets every edge in at most one point/. Define $\Gamma/x/ = \{E_i \setminus \{x\}: x \in E_i \in \mathcal{E}\}$ and $\Gamma/A/ = \bigcup_{x \in A} \Gamma/x/$. The following statement is proved in [4].

<u>Lemma</u> If A is a strong stable set in a τ-critical hypergraph then $|A| \leq |\Gamma/A/| - |\Gamma/x/| + 1$.
<u>Corollary</u> $|A| \leq |\Gamma/A/|$.

Since $V = A \cup T_o$ and $|T_o| \leq \tau \binom{\tau+r-2}{r-2}$, we have to show $|A| \leq \tau^{r-1}$. Instead, we will show $|\Gamma/A/| \leq \tau^{r-1}$ by giving a structure on T_o. Better to say, we give some sequences $x_1, \dots, x_{r-1}$ on T_o such that every edge E_i contains at least one of them.

In the beginning consider the elements of a fixed $T \in T_o$ as sequences of length one. Suppose that the /at most/ τ^k sequences of length k have been constructed. /$k \leq r-2$./ We define at most τ k+1-element sequences for each of them as follows:

Let $E_i \supset \{x_1, \dots, x_k\}$. Since $\tau_i \geq$ r-1, there exists a $T_j \in \mathcal{T}$ disjoint from $\{x_1, \ldots, x_k\}$. Adding any of the τ points of T_j to the set as x_{k+1}, we obtain τ sequences of length k+1. /If there is no edge containing $x_1, \ldots, x_k$ then we delete this sequence./ Obviously, $|\Gamma/A/|$ is not greater than the number of sequences of length r-1 that is at most τ^{r-1}.

References

[1] C.Berge: Graphs and hypergraphs, North Holland 1973.

[2] B.Bollobás: On generalized graphs, Acta Math. Acad. Sci. Hung. 16./1965/ 447-452.

[3] P.Erdős and T.Gallai: On the maximal number of vertices representing the edges of a graph, MTA Mat. Kut. Int. Közl. 6./1961/ 181-203.

[4] A.Gyárfás, J.Lehel and Zs.Tuza: Upper bound on the order of τ-critical hypergraphs, to appear

[5] G.O.H.Katona: Extremal problems for hypergraphs, Combinatorics /ed. M.Hall and J.H.Van Lint/ Mathematical Centre Tracts, 56./1974/ 13-42.

[6] Gy.Petruska and E.Szemerédi: On a combinatorial problem I., Studia Sci. Math. Hung., 7. /1972/ 363-374.