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On flatness and some ergodic super-properties of Banach spaces

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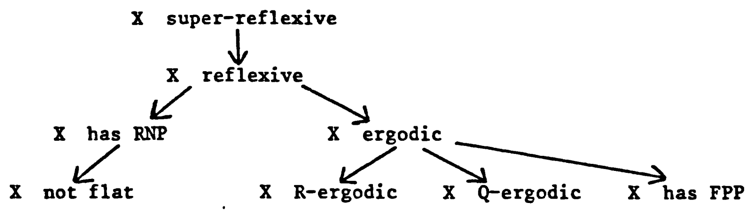
Let X be a Banach space. The following implications are well-known:



Recall that X is called super-P iff, for each Banach space Y, $Y \prec X$ implies that Y is P, where $Y \prec X$ means that for each finite-dimensional subspace F of Y and for each $\varepsilon > 0$ there is a subspace F' of X with $d(F,F') < 1 + \varepsilon$.

The following results were known already:

(1) X is super-reflexive iff X is super-non-flat. [3]

(2) X is super-reflexive iff X is super-R-ergodic. [1]

(3) X is super-reflexive iff X is super-Q-ergodic. [2]

Now we unify these results:

<u>Theorem 1</u> [5]. A Banach space X is not super-reflexive iff there is a Banach space Y with $Y \prec X$, Y completely flat, Y not R-ergodic, Y not Q-ergodic, and Y fails to have the FPP.

Let us first define some of the used concepts.

<u>Definition 2.</u> A Banach space X is <u>ergodic</u> (for isometries) iff for every isometry $T : X \to X$ and for every $x \in X$, the sequence $\left(\frac{1}{n} \sum_{i=1}^{n} T^{i-1} x\right)_{n=1}^{\infty}$ converges.

Definition 3. A real infinite matrix $(\rho_{i,j})_{i,j=1}^{\infty}$ is an R-matrix iff

(4) $\sum_{j=1}^{\infty} \rho_{i,j} \not\to 0$ if $i \to \infty$ and

(5) $\lim_{i\to\infty} \rho_{i,j} = 0$ for each $j \in \mathbb{N}$

Condition (4) means that $\sum_{j=1}^{\infty} \rho_{i,j}$ exists for each $i \in \mathbb{N}$, and the sequence $\left(\sum_{j=1}^{\infty} \rho_{i,j}\right)_{i=1}^{\infty}$ diverges or converges to a limit $\neq 0$.

Definition 4. A Banach space X is R-ergodic (for isometries) iff for each isometry $T : X \to X$ and for each $x \in X$, there is an R-matrix $(\rho_{i,j})_{i,j=1}^{\infty}$ such that $\left(\sum_{j=1}^{\infty} \rho_{i,j} T^{j-1}x\right)_{i=1}^{\infty}$ converges weakly.

To avoid too much technicalities we skip the definition of Q-ergodicity (see [2] , [5]).

Definition 5. A Banach space X has the FPP (fixed point property) (for isometries) iff for each isometry $T : X \to X$ and for each closed bounded convex $K \subset X$ with $TK \subset K$, there is an $x \in K$ with $Tx = x$.

Definition 6. A Banach space X is flat if there is a function $g : [0,1] \to \{x \in X : \|x\| = 1\}$ (called girth curve) with $g(0) = -g(1)$ and g is Lipschitz continuous with constant 2.
If $X = \overline{\text{span}}\, \{g(t) : 0 \le t \le 1\}$ then X is called completely flat.

Example 7. $L^1[0,1]$ is completely flat. The function $j : t \mapsto -\chi_{[0,t)} + \chi_{[t,1]}$ is a girth curve.

The importance of this example is demonstrated by the following

Theorem 8 [4]. A Banach space X is completely flat iff X is (isometric to) the completion of $L^1[0,1]$ for a norm $\|\cdot\|_X$ with for all $f \in L^1[0,1]$

(6) $$\|f\|_{L^1[0,1]} \ge \|f\|_X \ge \sup\{|\langle j^*(t), f\rangle| : 0 \le t \le 1\},$$

(where $j^*(t) = -\chi_{[0,t)} + \chi_{[t,1]} \in L^{\infty}[0,1]$), and then the function j (as in Example 7) is a spanning girth curve.

Now we'll give an idea of the proof of Theorem 1.

First note that $L^1[0,1]$ has almost all the properties that Y in Theorem 1 should have. Indeed, $L^1[0,1]$ is completely flat, and the isometry $T : L^1[0,1] \to L^1[0,1]$ defined by

$$(7) \qquad T\,f\,(t) = \begin{cases} 2f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

can be used to show that $L^1[0,1]$ is not R-ergodic, not Q-ergodic, and fails to have the FPP.

E.g., if $L^1[0,1]$ would be R-ergodic, there was an R-matrix $(\rho_{i,j})_{i,j=1}^{\infty}$ and an $f_0 \in L^1[0,1]$ with $\text{w-}\lim_{i\to\infty} \sum_{j=1}^{\infty} \rho_{i,j}\, T^{j-1}\, \chi_{[0,1]} = f_0$.
Then $< j*(0), f_0 > = \lim_{i\to\infty} < j*(0), \sum_{j=1}^{\infty} \rho_{i,j}\, T^{j-1}\, \chi_{[0,1]} > = \lim_{i\to\infty} \sum_{j=1}^{\infty} \rho_{i,j}$.
But also $< j*(0), f_0 > = \lim_{n\to\infty} < j*(2^{-n}), f_0 >$, and for fixed n a simple calculation, using (5), shows that $< j*(2^{-n}), f_0 > = -\lim_{i\to\infty} \sum_{j=1}^{\infty} \rho_{i,j}$, a contradiction by (4).

Also, if $f_0 \in K = \overline{\text{co}}\,\{2^n \chi_{[0,2^{-n}]} : n = 0,1,2,\dots\} \subset L^1[0,1]$ is arbitrary, then for any $\varepsilon > 0$ there is a convex combination f of $2^n\chi_{[0,2^{-n}]}$ $(n = 0,1,2,\dots)$ with $\| f_0 - f\| < \varepsilon$, and a $\delta > 0$ with $< j*(\delta), f > \; > 1 - \varepsilon$. But for $2^{-m} < \delta$ we have $< j*(\delta), T^m f > = -1$, so $\| f_0 - T^m f_0\| > 1 - 3\varepsilon$, and therefore $f_0 \ne Tf_0$.
So $L^1[0,1]$ doesn't have the FPP.

Now let Y be an arbitrary completely flat space, and identify Y with a completion of $L^1[0,1]$ according to Theorem 8. If T as defined by (7) can be extended to an isometry on Y, exactly the same reasoning as for $L^1[0,1]$ gives that Y is not R-ergodic, not Q-ergodic, and fails to have the FPP.

So for the proof of Theorem 1, if X is not super-reflexive, we want to find a Y as above, with $Y \prec X$. Note that, by (1), we may assume that X is completely flat. (If $Y \prec Z_1 \subset Z \prec X$, with Z_1 completely flat, then $Y \prec X$.)

To construct Y, we'll make a new norm $\| \cdot \|_0$ on the subspace Y_0 of $L^1[0,1]$ defined by $Y_0 = \text{span} \bigcup_{n=1}^{\infty} L^1[2^{-n}, 2^{-n+1}]$ (We consider $L^1[2^{-n}, 2^{-n+1}]$

as the subspace of $L^1[0,1]$ consisting of the functions with value 0 a.e. outside of $[2^{-n}, 2^{-n+1}]$.) , and then take for Y the completion of $(Y_0 , \|\cdot\|_0)$.

Take $y \in Y_0$. Then there are $n_0 \in \mathbb{N}$ and $y_n \in L^1[2^{-n}, 2^{-n+1}]$ $(n = 1,\dots,n_0)$ with $y = \sum_{n=1}^{n_0} y_n$. If $S = \{s_1,\dots,s_{n_0}\}$ is a subset of $\mathbb{N}$ with $s_1 < s_2 < \dots < s_{n_0}$, define $y_S = \sum_{n=1}^{n_0} T_{S,n}\, y_n$, with $T_{S,n}$ the natural isometry from $L^1[2^{-n},2^{-n+1}]$ onto $L^1[2^{-s_n},2^{-s_n+1}]$. Now the following holds:

(8) $\begin{cases} \text{There is a subsequence } \mathbb{N}_0 \text{ of } \mathbb{N} \text{ such that for all} \\ y = \sum_{n=1}^{n_0} y_n \in Y_0 \text{ there is a number } \|y\|_0 \in \mathbb{R} \text{ such that for all} \\ \varepsilon > 0 \text{ there is an } n(\varepsilon) \in \mathbb{N} \text{ with } \big| \|y\|_0 - \|y_S\|_X \big| < \varepsilon \text{ when-} \\ \text{ever } S = \{s_1,\dots,s_{n_0}\} \subset \mathbb{N} \text{ and } n(\varepsilon) < s_1 < \dots < s_{n_0} . \end{cases}$

(For the proof of (8), for a fixed $y \in Y_0$ apply Ramsey's theorem countably many times and use a diagonal procedure, and then repeat this for all y in a countable $L^1[0,1]$-dense subset of Y_0 , and again diagonalize.)

With y and S as above, it is easy to see, using (6), that $\|y_S\|_X \le \|y\|_{L^1[0,1]}$ and that for any t there is a t_S with

$|\langle j^*(t), y\rangle| = |\langle j^*(t_S), y_S\rangle| \le \|y_S\|_X$, so

(9) $\|y\|_{L^1[0,1]} \ge \|y\|_0 \ge \sup\{|\langle j^*(t),y\rangle| : 0 \le t \le 1\} \quad (y \in Y_0)$.

Now it is not hard to see that $\|\cdot\|_0$ is a norm on Y_0 , that the completion Y of $(Y_0 , \|\cdot\|_0)$ is completely flat by Theorem 8, and that T as defined by (7) can be extended to an isometry on Y.

To complete the proof of Theorem 1, we have to show that

(10) $Y \prec X$.

So let F be a finite-dimensional subspace of Y , and take $\varepsilon > 0$. Then there is a subspace F_0 of Y_0 with $d(F,F_0) < 1 + \varepsilon$. Take a finite ε-net for $\|\cdot\|_{L^1[0,1]}$ in $\{y \in F_0 : \|y\|_0 = 1\}$. Then there is an $n(\varepsilon)$ satisfying (8) for all y in this ε-net , and we can take $S \subset \mathbb{N}_0 \setminus\{1,\dots,n(\varepsilon)\}$ such that with $F_1 = \{y_S : y \in F_0\} \subset X$ we get $d(F_0 , F_1) < \dfrac{1 + 2\varepsilon}{1 - 2\varepsilon}$.

Remark 9. If we replace the requirement 'Y completely flat' in Theorem 1 by 'Y flat', then we can add 'Y doesn't have the Krein-Milman Property'. Indeed, take Y^{**} instead of Y. Then Y^{**} is flat (but not completely flat), so it doesn't have the Radon-Nikodym Property, and as a dual space it doesn't have the KMP.

Question 10. If X is flat, does there exist a Y with $Y \prec X$, Y completely flat, Y doesn't have the KMP?
Or, does flatness already exclude the KMP?

Question 11. Does flatness already exclude R-ergodicity and/or Q-ergodicity and/or the FPP, in other words, does Theorem 1 say anything more than result (1)?

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