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TRANSVERSE G-STRUCTURES ON FOLIATED MANIFOLDS

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Let M be a compact connected n-dimensional manifold endowed with a q-codimension foliation $\mathcal{F}$. All the structures are assumed to be C^∞.

1. Transverse fields ; transverse G-structures

We denote by P the subbundle of TM tangent to the leaves of the foliation. $Q = TM/P$ is the transverse bundle of $(M,\mathcal{F})$.

If X is a foliated vector field, it defines a section $\bar{X}$ of Q. $\bar{X}$ is the transverse field associated to X. The set $\ell(M,\mathcal{F})$ of transverse fields has a natural Lie algebra structure.

We denote by $E_T(M,p_T,GL(q,R))$ the principal bundle of frames of Q. E_T is the bundle of transverse frames of $(M, \mathcal{F})$. It is endowed with a natural structure form θ_T, which is a R^q-valued tensorial form. Using θ_T, we define on E_T a lifted foliation $\mathcal{F}_T$ in the following way : $X_z \in T_zE_T$ is tangent to the leaf of $\mathcal{F}_T$ at z iff $i_{X_z}\theta_{T_z} = i_{X_z}d\theta_{T_z} = 0$. We denote by P_T the subbundle of TE_T tangent to the leaves of the lifted foliation.

If $e_T(M,p_T,G)$ is a G-subbundle of E_T such that

$$P_{T_z} \subset T_z(e_T) \qquad \forall z \in e_T ,$$

e_T is a transverse [or foliated] G-structure on $(M,\mathcal{F})$ [1] [3] [4].

2. Transverse parallelisms ; Lie foliations.

If $G = \{e\}$, a transverse G-structure on $(M,\mathcal{F})$ is a transverse parallelism [1] [5]. Such a structure is determined by q transverse fields $\{\bar{X}_1,\ldots,\bar{X}_q\}$ which are linearly independant at each point. In this case, we say that $(M,\mathcal{F})$ is a parallelisable foliation.

If, moreover, $\{\bar{X}_1,\ldots,\bar{X}_q\}$ is a basis of a q-dimensional Lie subalgebra $\mathfrak{g}$ of $\mathcal{L}(M,\mathcal{F})$, one says that $(M,\mathcal{F})$ has a structure of Lie $\mathfrak{g}$-foliation. Lie foliations have been studied by Fedida in [2]. In [7], we obtained

Theorem 1. If $(M,\mathcal{F})$ is a parallelisable foliation, then the closures of the leaves are the fibers of a fibration $\pi : M \to W$. Moreover, there exists a Lie algebra $\mathfrak{g}$ such that $\mathcal{F}$ induces on each fiber of π a Lie $\mathfrak{g}$-foliation.

First part of this result may be also obtained from a theorem of Sussmann [8].

If $\pi_1(M) = 0$, the structural Lie algebra $\mathfrak{g}$ is abelian. Using a well-known result of Tischler, the fiber of π admits, in this case, a fibration on T^r, where $r = q - \dim W$. From these remarks, we deduce

Theorem 2. If M is a simply connected compact manifold, M admits no 4-codimension parallelisable foliation.

3. Riemannian foliations

If $G = O(q)$, a transverse G-structure on $(M,\mathcal{F})$ is a transverse riemannian structure. We say that $(M,\mathcal{F})$, endowed with such a structure e_T, is a riemannian foliation. Riemannian foliations were introduced by B. Reinhart [7].

It is possible, in this case, to introduce a transverse Levi-Civita connection ω_T on e_T. Moreover, $\omega_T + \theta_T$ defines a transverse parallelism on $(e_T, \mathcal{F}_T)$. This fact allows us to use results of the previous section in order to study riemannian foliations. Several results are obtained in this way ; for example

<u>Theorem 3</u>. If $(M,\mathcal{F})$ is a riemannian foliation, and $\pi_1(M) = 0$, then there exists an algebra of transverse fields in the center of $\ell(M,\bar{\mathcal{F}})$ whose transverse orbits define the closures of the leaves.

Using the same methods, riemannian foliations are classified in [6] in codimension ≤ 3.

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