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1. Definitions

A *hypergraph* H is a set of different non-empty subsets called edges chosen from some finite basic set. The edge set of H is denoted by $E(H)$ and the basic set called vertex set of H is denoted by $V(H)$.

The *section hypergraph* induced by a set $A \subseteq V(H)$ is a hypergraph with vertex set A and edge set $\{e \in E(H) \mid e \subseteq A\}$. The *partial hypergraph* induced by a set $B \subseteq E(H)$ is a hypergraph with edge set B and the elements of the edges form its vertex set.

We introduce the following notations:

$\alpha(H)$ = *weak stability number* = maximum cardinality of a weakly stable set of H = maximum number of vertices containing no edge of H;

$\rho(H)$ = *covering number* = minimum number of edges and vertices whose union is $V(H)$;

$\rho_0(H)$ = *partition number* = minimum number of pairwise disjoint edges and vertices with union $V(H)$;

$\nu(H)$ = packing number = maximum number of pairwise disjoint edges of H;

$\tau(H)$ = transversal number = minimum number of vertices meeting all edges of H.

A hypergraph is said to be r-uniform if its edges contain just r vertices. An r-uniform hypergraph with vertex set $\mathcal{V}$ will be called complete if every r-tuples of $\mathcal{V}$ is an edge. The edge set $\mathcal{E}$ of an r-uniform hypergraph is called K_p-free if no subset of $\mathcal{E}$ generates K_p the r-uniform complete hypergraph of order p.

The set $\{\mathcal{E}_i\}_{i=1}^{k}$ is called a K_p-cover of the r-uniform hypergraph $(\mathcal{V},\mathcal{E})$ if $\bigcup_{i=1}^{k}\mathcal{E}_i = \mathcal{E}$ and every $\mathcal{E}_i$ is a K_p or an edge; k will be called the size of this K_p-cover. A K_p-cover with pairwise disjoint elements is said to be a K_p-partition.

Let F be a given r-uniform hypergraph. The F-hypergraph H/F of an r-uniform hypergraph H is defined by $V(H/F) = E(H)$ and $E(H/F) = \{F' \subset E(H) \mid F' \cong F\}$.

2. General results

We give a survey of some recent results concerning various relations between hypergraph numbers defined above.

Theorem 1. Every r-uniform hypergraph H satisfies:

$$\rho_v(H) + (r-1)\cdot\nu(H) = |V(H)| = \alpha(H) + \tau(H) .$$

By definition $\rho(H) \leqq \rho_o(H)$, however, in case of graphs $\rho_o = \rho$. Thus Theorem 1 has the following corollary:

Theorem (T.Gallai [2]). Every simple graph G satisfies:

$$\rho(G) + \nu(G) = \alpha(G) + \tau(G) \ .$$

The next result is a possible hypergraph extension of the well-known König's theorem, however, its proof (see in [3]) uses the König-Hall theorem itself.

Theorem 2. If $\alpha(H') \geq \frac{1}{2} \cdot |V(H')|$ holds for every section hypergraph H' of H then $\rho(H) \leqq \alpha(H)$.

It is worth to note that the next two statement are trivially equivalent:

(i) $\alpha(H') \geqq \frac{1}{2} \cdot |V(H')|$ for every section hypergraph H' of H ;

(ii) $\alpha(H') \geqq \frac{1}{2} \cdot |V(H')|$ for every partial hypergraph H' of H .

Thus the condition in Theorem 2 may be replaced by (ii) without any consequence.

The analogous problem of describing non-trivial hypergraph classes with $\rho_o \leqq \alpha$ seems to be a rather hard question. Perhaps the first instance of this problem was the well-known Ryser conjecture:

If the vertex set of the r-uniform hypergraph H is partitionned into r classes so that every edge contains just one vertex from each classes, i.e.. H is an r-partite hypergraph, then $\tau(H) \leqq (r-1) \cdot \nu(H)$.

By Theorem 1 $\tau \leqq (r-1) \cdot \nu$ iff $\rho_o \leqq \alpha$ therefore Ryser's conjecture may be restated in the following form:

<u>Conjecture.</u> r-partite hypergraphs satisfy $\rho_0 \leq \alpha$.

3. Hypergraphs with $\rho \leq \alpha$

We present here hypergraph classes with property (i). By Theorem 2 the hypergraphs belonging to theses classes will satisfy $\rho \leq \alpha$.

The 2-coloration of a hypergraph H is a partition of V(H) into two weakly stable sets immediately implying property (i) :

<u>Theorem 3.</u> 2-colorable hypergraphs satisfy $\rho \leq \alpha$.

The next observation certainly belongs to the folklore of extremal graph theory:

More than the half of the edges of an arbitrary graph can be retained to form a bipartite partial graph.

This observation has the following consequence (see in [4]):

<u>Theorem 4.</u> If F is a graph with chromatic number greater than 2 then for every graph G the F-hypergraph of G satisfies $\rho(G/F) \leq \alpha(G/F)$.

The observation above can be extended for hypergraphs (see in [3]) which yields our next result:

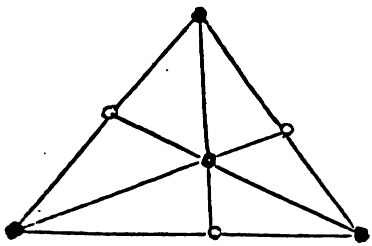
<u>Theorem 5.</u> For any $1 < r < p$ the K_p-hypergraph of every r-uniform hypergraph H satisfies $\rho(H/K_p) \leq \alpha(H/K_p)$.

Remark that by the definitions $\rho(H/K_p)$ is the minimum size of a K_p-cover of H and $\alpha(H/K_p)$ is the maximum cardinality of a K_p-free edge set of H.

Theorem 5 answers a conjecture of B.Bollobás [1] :

<u>Corollary</u>. The edge set of every r-uniform hypergraph of order n can be covered with at most $T(n,p,r)$ K_p's and edges where $T(n,p,r)$ is the <u>extended Turán number</u>, i.e., the maximal number of edges an r-uniform hypergraph of order n can have if it does not contain a K_p.

Theorem 5 may suggest the question whether the class of K_p-hypergraphs satisfy the stronger $\varrho_0 \leq \alpha$. The answer is not known even if $r=2$ and $p=3$ except some special cases settled by Zs.Tuza. Let's remark finally that the analogous question on 2-colorable hypergraphs has a negative answer; P_r^- the r-uniform hypergraph of the finite projective plane minus one line may be an example which is clearly 2-colorable with weak stability number r^2-2r+1 smaller than the partition number r^2-2r+2.



P_3^-

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