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Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces

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The question we treat here can be formulated generally as follows: What can be said about the linear structure of uniformly or Lipschitz homeomorphic Banach spaces? Under which conditions are they isomorphic, have the same local structure, etc? From the extended list of results in this direction (a historical survey is given in [4]) we mention only two: By Enflo [2] , each Banach space uniformly homeomorphic to a Hilbert space is isomorphic to it. On the other hand, Aharoni and Lindenstrauss gave an example of two (nonseparable and nonreflexive) Lipschitz homeomorphic but not isomorphic Banach spaces [1] .

We show that despite of this example, under some natural conditions, Lipschitz homeomorphism implies linear isomorphism. This is achieved by the help of differentiation methods. Using model-theoretic techniques, we derive from these results new information about uniformly homeomorphic Banach spaces and give short proofs of the results of Ribe [9] , [10] .

1. Definitions

Let $f : X \longrightarrow Y$ be a mapping from a Banach space X into a Banach space Y. If f is a one-to-one surjection with f and f^{-1} uniformly continuous, f is called a uniform homeomorphism. We say that f is a Lipschitz mapping if there is a constant K such that for all $x, y \in X$,

$$\|f(x) - f(y)\| \leq K \|x - y\| .$$

The mapping f is a Lipschitz embedding if f is a one-to-one mapping from X into Y such that f and $f^{-1}\big|_{f(X)}$ are Lipschitz. Lipschitz homeomorphism means surjective Lipschitz embedding. Finally, we say that a set $A \subset X$ is the range of a Lipschitz projection from X if there is a Lipschitz map from X onto A which is the identity on A. On the other hand, the words „isomorphic" or „complemented" always stand for „linearly isomorphic" and „linearly complemented".

A mapping $f : X \rightarrow Y$ is said to be Gateaux differentiable at $x_0 \in X$ if for every $x \in X$ the limit

$$\lim_{\lambda \to 0} \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} = (Df)_{x_0}(x)$$

exists and the so-defined mapping $(Df)_{x_0} : X \rightarrow Y$ is linear.

Given a Banach space X and an ultrafilter U on some set I, we consider the l_∞-sum of I copies of X,

$$l_\infty(I,X) = \left\{ (x_i) : x_i \in X , \|(x_i)\| = \sup_I \|x_i\| < \infty \right\}$$

and its subspace

$$N_U = \left\{ (x_i) \in l_\infty(I,X) : \lim_U \|x_i\| = 0 \right\} .$$

The quotient $l_\infty(I,X)/N_U$ is called the ultrapower $(X)_U$ of the space X with respect to U. For some background on ultrapowers see [3].

2. Lipschitz homeomorphic spaces

It is easily seen that (once it exists) the differential of a Lipschitz embedding is an isomorphic embedding. The main observation is the following

Proposition 1. Let X and Y be Banach spaces and assume that X is complemented in X^{**}. Let f be a Lipschitz embedding of X into Y which is Gateaux differentiable at some point $x \in X$. If $f(X)$ is the range of a Lipschitz projection from Y, then the image of the differential, $(Df)_x(X)$, is a complemented subspace of Y.

A Lipschitz embedding of a separable Banach space into a reflexive space possesses a Gateaux differential at sufficiently many points [7]. From this we get

Theorem 2. Let f be a Lipschitz embedding of a separable Banach space X into a reflexive Banach space Y, and assume that $f(X)$ is the range of a Lipschitz projection in Y. Then X is isomorphic to a complemented subspace of Y.

With this theorem, Pełczyński's decomposition method ([8], [5]) enables us to derive the following classification results.

Corollary 3. Let X and Y be separable reflexive spaces, each of them isomorphic to its Cartesian square. If X and Y are Lipschitz homeomorphic, they are isomorphic.

Problem 4. Are every two separable, reflexive Lipschitz homeomorphic Banach spaces isomorphic?

Corollary 5. Let $p \in (1,\infty)$ and let X be isomorphic to either $L_p[0,1]$ or l_p. Then each Banach space Lipschitz homeomorphic to X is isomorphic to X. The same is true if X is a superreflexive rearrangement invariant function space on $[0,1]$, e.g. an Orlicz space $L_M[0,1]$ with M being a reflexive Orlicz function.

Lipschitz mappings into spaces without the Radon-Nikodým property may not possess differentials at all. Therefore the question arises what can be said about the general case. Using a weaker form of differentiation (essentially:differentiating the scalar-valued function $\langle f(.)\ ,\ y^*\rangle$ for sufficiently many $y^* \in Y^*$) and the Loewenheim-Skolem Theorem, we can show

Theorem 6. Let X be Lipschitz homeomorphic to Y. Then every separable subspace of X embeds isomorphically into Y^{**}.

3. Uniformly homeomorphic spaces

The connection with Lipschitz mappings is established via ultrapowers:

Proposition 7. Let X and Y be uniformly homeomorphic Banach spaces and let U be a non-trivial ultrafilter on $\mathbb{N}$. Then $(X)_U$ and $(Y)_U$ are Lipschitz homeomorphic.

Immediately from Proposition 7, Theorem 6 and the principle of local reflexivity we get the following result, due to Ribe [9].

Theorem 8. Let X be uniformly homeomorphic to Y. Then there exists a constant K such that each finite dimensional subspace of X is K-isomorphically embeddable into Y.

If we assume that Y is superreflexive, then its ultrapower $(Y)_U$ is reflexive, which puts us into a position to apply Theorem 2 (modulo some model-theoretic techniques to guarantee the separability assumption). This leads to

Proposition 9. If X is uniformly homeomorphic to a superreflexive space Y, then there exists an ultrafilter U such that X is isomorphic to a complemented subspace of $(Y)_U$.

The proof of the next result involves the Keisler-Shelah Isomorphism Theorem.

Theorem 10. Let X and Y be Banach spaces which are isomorphic to its Cartesian square, and assume that Y is superreflexive. If X and Y are uniformly homeomorphic, then there exists an ultrafilter U such that $(X)_U$ and $(Y)_U$ are isomorphic.

Naturally, it arises

Problem 11. Do every two uniformly homeomorphic Banach spaces have isomorphic ultrapowers?

Note that the two spaces found by Aharoni and Lindenstrauss [1] do so.

Another result of Ribe [10] can be obtained as a consequence of Proposition 9.

Theorem 12. Let $p \in (1, \infty)$. If X is uniformly homeomorphic to an $\mathcal{L}_p$ space, then X itself is an $\mathcal{L}_p$ space.

Using a result of Lindenstrauss [6] and the techniques presented above, we can solve one of the remaining cases for p.

Theorem 13. If X is uniformly homeomorphic to an $\mathcal{L}_\infty$ space, then X is an $\mathcal{L}_\infty$ space.

It remains open

Problem 14. Does Theorem 12 hold for $p = 1$?

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