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DENSITY THEOREMS FOR MEASURABLE TRANSFORMATIONS

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1. Introduction

Let m denote Lebesgue measure on Borel σ-algebra of the unit interval $[0,1]$. A function $\tau:[0,1]\to[0,1]$ which is Borel-measurable and nonsingular (i.e. $m(A)=0 \Rightarrow m\tau^{-1}(A)=0$) is called a transformation. We identify transformations which differ only on a set of measure zero. A transformation τ is called invertible if τ^{-1} exists and is also a transformation. τ is called measure-preserving if $m\tau^{-1}(A)=m(A)$ for all Borel A. The group of all invertible transformations is denoted by G. By G_m we denote the group of all invertible measure- preserving transformations.

Every invertible transformation τ induces a positive invertible isometry $T_\tau^{(p)}$ of $L^p(m)$, $1\le p<\infty$, defined by

$$T_\tau^{(p)}(f)(t)=\omega_\tau^{1/p}(t)\ f(\tau^{-1}(t)) \quad ,$$

where $f\in L^p(m)$, $\omega_\tau = dm\tau^{-1}/dm$. If $\tau\in G_m$, then $\omega_\tau=1$.

By a classical result (see e.g. Ionescu Tulcea [2], footnote 3), for every $1\le p<\infty$ we can identify G with the group $G^{(p)}$ of all positive invertible isometries of $L^p(m)$ (i.e. with the set of all Banach lattice automorphisms of $L^p(m)$). Therefore we can define a topology in G as the strong operator topology inherited from $L(L^p(m))$. For all $1 \leq p < \infty$ these topologies coincide (Choksi, Kakutani [1], Theorem 8). Moreover, L^p - strong and weak operator topologies in G_m coincide, since the strong and weak topologies on the unitary group in $L(L^2(m))$ are the same and all L^p - weak operator topologies coincide on the compact set of doubly stochastic operators. It is not hard to see that the family of sets of the form

$$\{\tau \in G: m(\tau(A_i) \triangle \sigma(A_i)) < \varepsilon \text{ for } i=1, \dots, n \text{ and } \|\omega_\tau - \omega_\sigma\|_1 < \varepsilon\}$$

where $\varepsilon > 0$, $\sigma \in G$ and $A_1, \dots, A_n$ is a partition of $[0,1]$ into subintervals is a neighborhood base for the strong operator topology in G .

In this papier we prove that the group G_m and G are topologically finitely generated (Theorem 1 and 2).

2. Invertible measure-preserving transformations.

We will use following property of permutations

Lemma 1. Let n be a natural number. The group of all permutations of $\{1, \dots ,2n\}$ is generated by the following two elements

$$\alpha=\begin{pmatrix} 1,2, & \dots & ,2n \\ 2,3, & \dots & ,1 \end{pmatrix}, \quad \beta=\begin{pmatrix} 1,2, & \dots,n-1,n,n+1, & \dots & ,2n \\ 2,3, & \dots, n ,1,n+1, & \dots & ,2n \end{pmatrix}$$

Proof. Since every permutation can be decomposed into transpositions , it suffies to show that α and β generate every transposition . Moreover, because of the nature of α and β it is enough to prove that some transposition, e.g.

$$\gamma=\begin{pmatrix} 1,2, & \dots, n,n+1, & \dots & ,2n \\ 1,2, & \dots,2n,n+1, & \dots & , n \end{pmatrix}$$

can be expressed as a composition of α and β . In fact it is not hard to see that $\gamma=\alpha^{n-1}\beta\alpha^{n}\beta$.

For $a\in[0,1]$ we write $\alpha_a(t)= t+a \pmod 1$.

Moreover, we define

$$\beta_a(t)=\begin{cases} t+a \mod 1/2 & \text{for } 0\leq t<1/2 \\ t & \text{for } 1/2\leq t\leq 1 \end{cases} .$$

Obviously $\alpha_a ,\beta_a \in G_m$.

Theorem 1. Let a and b be irrational numbers . Then the group generated by α_a and β_b is dense in G_m .

Proof. It is easy to see that for every real number c,the transformations α_c and β_c belong to the closure $\mathcal{H}_m$ in G_m of the group generated by α_a and β_b .

Now given $n \in \mathbb{N}$, we partition $[0,1]$ into $2n$ subintervals of equal length . It is sufficient to show that $\mathcal{H}_m$ contains every piecewise linear transformation ξ which permutes these subintervals . From Lemma 1 we can express ξ as a certain composition of transformations α_c and β_c for $c=1/2n$.

3. Invertible transformations .

Let $I_1, \dots, I_n$ and $J_1, \dots, J_n$ be partitions of the interval $[0,1]$ into subintervals. The notation $\varphi : I_1 \to J_1 , \dots , I_n \to J_n$ will mean that φ is the piecewise linear transformation that maps I_i linearly (with positive slope) onto J_i for all $i \leq n$.

Now for $a \in [0,1]$ and $b > 0$ we define $\varphi_{a,b}$: $[0, a/(b+1)] \to [0, ab/(b+1)]$, $(a/(b+1), a] \to (ab/(b+1), a]$, $(a,1] \to (a,1]$. Note that the first interval is stretched and the second is shrunk by the factor of b .

Let $\mathcal{H}$ denote the group generated by G_m and ψ where $\psi = \varphi_{1/4,2} \circ \alpha_{1/2} \circ \varphi_{1/4,3} \circ \alpha_{1/2}$: $[0,1/12) \to [0,1/6)$, $[1/12,1/4) \to [1/6,1/4)$, $[1/4,1/2) \to [1/4,1/2)$, $[1/2, 9/16) \to [1/2, 11/16)$, $[9/16 , 3/4) \to [11/16 , 3/4)$, $[3/4,1] \to [3/4,1]$.

Lemma 2. Let $a\in[0,1]$. Then $\varphi_{a,2}$, $\varphi_{a,3}$ belong to $\mathcal{H}$.

Proof. We may assume that $a \leq 1/12$ (since $G_m \subset \mathcal{H}$, several conjugates of φ can be composed together, if necessary). Let $\xi \in G_m$ be defined by ξ:
$[0, 1/6) \to [1/12+ 2a), [2a+ 1/4, 1/6, 2a+1/4) \to [0, 2a+1/12)$,
$[2a+ 1/4, 1/2) \to [2a+ 1/4, 1/2), [1/2, 11/16) \to [9/16, 3/4)$,
$[11/16, 3/4) \to [1/2, 9/16)$, $[3/4, 1] \to [3/4, 1]$.
The transformation $\varphi = \psi \xi \psi$ transforms linear intervals $I_1 = (1/12 - a, 1/12)$ and $I_2 = (1/4, 1/4 + 2a)$ onto $(1/4, 2a+ 1/4)$ and $(1/6, a+ 1/6)$, respectively. It is easy to check that for all intervals I with $I \cap (I_1 \cup I_2) = \emptyset$ we have $m(\varphi(I)) = m(I)$. This implies the existence of two transformations $\xi_1, \xi_2 \in G_m$ such that $\varphi_{a,2} = \xi_1 \varphi \xi_2$. Therefore we obtain $\varphi_{a,2} \in \mathcal{H}$.

By analoguous arguments, $\varphi_{a,3} \in \mathcal{H}$.

Lemma 3. If $\varphi_{a,b}$, $\varphi_{a,c} \in \mathcal{H}$ for all $a \in [0,1]$ and some $b,c > 0$, then $\varphi_{a,bc} \in \mathcal{H}$ for all $a \in [0,1]$.

Proof. Let $\delta > 0$ be such that $\delta(b+1)(c+1) \leq 1/2$. We put $\eta = \varphi_{b+1,b} \circ \alpha_{1/2} \circ \varphi_{b\,c+1,c} \circ \alpha_{1/2}$:
$[0, \delta] \to [0, \delta b]$, $(\delta, \delta(b+1)] \to (\delta b, \delta(b+1)]$,
$(\delta(b+1), 1/2] \to (\delta(b+1), 1/2]$, $(1/2, \delta b + 1/2] \to (1/2, \delta bc + 1/2]$,

$(\delta b + 1/2\,,\ b(c+1) + 1/2] \to (\delta bc + 1/2\,,\ b(c+1) + 1/2]$,

$(\delta b(c+1) + 1/2\,,\ 1] \to (\delta b(c+1) + 1/2\,, 1]$. We have .

Let now $\xi \in G_m$ be defined by $\xi : [0, \delta b) \to [1/2, \delta b + 1/2)$,

$[\delta b, \delta(b+1)) \to [0, \delta)$, $[\delta(b+1)\,,\ 1/2) \to [\delta(b+1), 1/2)$,

$[1/2\,,\ \delta bc + 1/2) \to [\delta b + 1/2, \delta b(c+1) + 1/2)$,

$[\delta bc + 1/2\,, \delta b(c+1)) \to [\delta, \delta(b+1))$,

$[\delta b(c+1) + 1/2\,,\ 1] \to [\delta b(c+1) + 1/2\,,\ 1]$. The transformation $\varphi = \eta\circ\xi\circ\eta$ transforms $I_1 = (0, \delta]$ and $I_2 = (\delta b + 1/2\,, \delta b(c+1) + 1/2]$ onto $(1/2, \delta bc + 1/2]$ and $(\delta b, \delta(b+1)]$ respectively , and for intervals I with $I \cap (I_1 \cup I_2) = \emptyset$ we have $m(\eta\circ\xi\circ\eta(I)) = m(I)$. Hence $\varphi_{a,bc} = \sigma\varphi\tau$ for some $\sigma, \tau \in G_m$ and we obtain $\varphi_{a,bc} \in \mathcal{H}$.

Corollary. The closure of $\mathcal{H}$ contains all $\varphi_{a,b}$ for $0 \leq a \leq 1$ and $b > 0$.

Proof. The transformation $\varphi_{a,b}$ belongs to $\mathcal{H}$ if and only if $\varphi_{a,\ 1/b}$ belongs to $\mathcal{H}$ since G_m . Therefore using Lemma 2 and Lemma 3 we obtain that $\varphi_{a,b} \in \mathcal{H}$ for $b = 2^k / 3^m$ with $k, m \in \mathbb{N}$. Because the set $\{ 2^k / 3^m : k, m \in \mathbb{N} \}$ is dense in $\mathbb{R}_+$ the proof is complete.

The following Proposition is implicitly contained in [3] ; we omit the proof .

Proposition. Let D be a dense subset of $[0,1]$. Then the family of all invertible transformations τ of the form $\tau : I_1 \to J_1, \ldots, I_n \to J_n$, where (I_k) and (J_k) are partitions of $[0,1]$ into subintervals with endpoints in $D \cup \{0,1\}$, is dense in G .

Theorem 2. Let a,b be irrational numbers . Then the group generated by α_a , β_b and ψ is dense in G .

Proof. In view of Theorem 1 and Proposition , it is sufficient to show that for every partitions $0=a_0<a_1< \dots <a_{n+1}=1$ and $0=b_0<b_1<\dots<b_{n+1}=1$ with a_i, b_i of the form $2^k/3^m$ for $1\leq i\leq n$ there exists a transformation in $\mathcal{H}$ which takes $[a_i, a_{i+1})$ linearly onto $[b_i, b_{i+1})$ for i=1, ... ,n .

By the last corollary, we may assume that $a_n \leq 1/4$ and $b_n \leq 1/4$.

Now $\xi_1 = \varphi_{a_1+b_1}$, b_1 / a_1 maps $[0,a_1)$ onto $[0,b_1)$. The function $\varphi: [0,b_1) \to [0,b_1)$, $[b_1, \xi_1(a_2)) \to [b_1, b_2)$, $[\xi_1(a_2), \xi_1(a_2)+b_2-b_1\ ) \to [b_2, \xi_1(a_2)+b_2-b_1\ )$, $[\xi_1(a_2)+b_2-b_1\ , 1] \to [\xi_1(a_2)+b_2-b_1\ , 1]$ clearly satisfies $\varphi = \sigma\varphi_{x,y}\tau$ for some $\sigma,\tau \in \mathcal{G}_m$ and $x,y \in \mathbb{R}_+$ and so $\varphi\in\mathcal{H}$. Therefore $\xi_2 = \varphi\xi_1 \in \mathcal{H}$ and it is easy to see that ξ_2 takes $[a_i, a_{i+1})$ onto $[b_i, b_{i+1})$ for i=0,1 . Continuing this process by induction , we can construct a transformation $\xi_n \in \mathcal{H}$ such that ξ_n takes $[a_i, a_{i+1})$ onto $[b_i, b_{i+1})$ for i=0,1, ... ,n .

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