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Isometric theory of duality in Banach spaces

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I) Problems in isometric duality theory

Let X, Y be Banach spaces. Let us say that Y is a dual space if there exists a space Z such that Z' is isometric to Y. Let us say that X is unique predual of X' if any Banach space Z such that Z' is isometric to X', is isometric to X. The problems in isometric theory of duality are mainly the following ones:

1) Find conditions on X which ensure that X is a dual space.
2) Find conditions on X which ensure that X is unique predual of X'.
3) Study the properties of the norms on X', X'', ...
4) Inverse the ( )' functor = let $\varphi : X' \to X'$ an isometry. Does there exist $\varphi_0 : X \to X$ such that $\varphi_0' = \varphi$?

We shall answer these problems for some special classes of Banach spaces.

II) Existence of preduals

This lemma is an application of Hahn-Banach theorem.

Lemma 1. Let E be a Banach space, and $f \in E''$. The following assertions are equivalent:

1) $\forall u \in E$, $\|f+u\| \geq \|u\|$.
2) $\operatorname{Ker} f \cap E_1'$ is ω^*-dense in E_1'.

(where E_1' is the unit ball of E').

From this lemma we can deduce.

Theorem 2. If the norm of E is Fréchet-differentiable on a dense subset of E, or if E is separable and does not contain $l^1(\mathbb{N})$, then the following assertions are equivalent:

1) E is isometric to a dual space.

2) There is a contractive projection from E'' onto E .

If 1) - 2) are satisfied , then there is only one contractive projection from E'' onto E , and the predual of E is unique.

Example. E an Asplund space, for any equivalent norm.

III) Unicity of preduals

Definition 3. Let E be a Banach space, and $\{x_n\}_{n\in \mathbb{N}}$ be a sequence in E . The sequence $\{x_n\}$ is said to be weakly unconditionally convergent (w.u.c.) if we have

$$\sum_{n=1}^{+\infty} |t(x_n-x_{n-1})| < +\infty \quad \forall t\in E' .$$

This lemma has been obtained by M. Talagrand and myself

Lemma 4. Let E be a Banach space. Let $\{x_n\}$ be a sequence in E' which tends to O in $(E',\sigma(E',E))$. If the sequence $\{x_n\}$ is w.u.c., then one has $\lim_{n\to+\infty} x_n = 0$ in $(E',\sigma(E',G))$ for any predual G of E'.

From lemmas 1 and 4 we can deduce the following theorem, which asserts that numerous properties are sufficient to ensure that a Banach space is unique predual

Theorem 5. The Banach spaces E belonging to one of the following classes, and their subspaces, are unique predual of their dual for every equivalent norm. Moreover, every bijective isometry of E' is the adjoint of an isometry of E .

a) spaces with Radon-Nikodym property,

b) spaces whose dual does not contain $l^1(\mathbb{N})$,

c) B-convex spaces,

d) weakly K-analytic spaces such that $c_o(\mathbb{N})$ is not a quotient space of E ,

e) weakly sequentially complet Banach spaces, complemented in a Banach lattice,

f) $\mathcal{L}^1$-spaces,

g) preduals of von Neumann-algebras,

h) spaces with local unconditional structure which does not contain l_n^∞ uniformly.

IV) Properties of norms on dual spaces

This theorem extends an old result of Dixmier

Theorem 6. Let E be a non-reflexive Banach space, 1-complemented in E''. Then the unit sphere of $E^{(2n+1)}$ contains a simplex of dimension n .

Corollary 7. Let E be a non-reflexive Banach space. Then the unit sphere of $E^{(2n+2)}$ contains a simplex of dimension n .

For example, the unit sphere of $E^{(8)}$ contains a tetrahedron. Note that if E is a non-reflexive Banach space which is isometric to one of his duals, then corollary 7 proves that the unit sphere of E contains simplex of any dimension.

Let E be an Asplund space, and N an equivalent norm on E . Let us call $\mathcal{T}(N)$ the "tangent space of N ", that is the norm-closed linear subspace of E' generated by the differentials at the points of Frechet-differentiability of N . Let us say that $N_1 \sim N_2$ if $\mathcal{T}(N_1) = \mathcal{T}(N_2)$, and let $\mathcal{N}$ be the set of equivalence classes of norms on E for the relation $\sim$. The set $\mathcal{N}$ is ordered by

$$\dot{N}_1 \succ \dot{N}_2 \Longleftrightarrow \mathcal{T}(N_1) \supseteq \mathcal{T}(N_2) .$$

We have now

Theorem 8. Let E be an Asplund space, and N an equivalent norm on E . The following assertions are equivalent:

1) (E, N) is a dual space.
2) $\dot{N}$ is minimal in the ordered set $(\mathcal{N}, \succ)$.

In other words, it is necessary and sufficient, for N to be a dual norm, that N be "as less differentiable as possible" between the equivalent norms on E .