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Definition. A Hausdorff space X is said to be pseudobasically compact iff there exists a pseudobase $\mathcal{C}$ of X and a relation $<$ defined on $\mathcal{C}$ such that

(a) if $U, V \in \mathcal{C}$ and $U < V$, then $U \subseteq V$ and $U \neq V$,

(b) if $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{B}$ is a chain with respect to $<$, then $\bigcap \mathcal{B} \neq \emptyset$,

(c) for each open set $W \subseteq X$ and $V \in \mathcal{C}$ if $W \cap V \neq \emptyset$ then there exists $U \in \mathcal{C}$ such that $U \subseteq W$ and $U < V$.

The following two lemmas are just simple observations:

Lemma 1. An open subset of a pseudobasically compact space is pseudobasically compact.□

Lemma 2. The closure of an open subset of a pseudobasically compact space is pseudobasically compact.□

The following is not so trivial:

Lemma 3. A dense G_δ set of a pseudobasically compact space is pseudobasically compact.

Proof. Let X be a pseudobasically compact space and let $\{U_n : n=1,2,\dots\}$ be a decreasing sequence of open sets of X such that $G = \bigcap\{U_n : n=1,2,\dots\}$ is dense. Let $\mathcal{C}$ be a pseudobase of X and let (a)-(c) hold for $\mathcal{C}$. Consider families $\mathcal{C}_0 = \{U \in \mathcal{C} : U \subseteq \operatorname{Int} G\}$ and $\mathcal{C}_n = \{U \cap G : U \in \mathcal{C}$ and $U \subseteq U_n \setminus \operatorname{cl}\operatorname{Int} G\}$

for $n=1,2,\ldots$. Put $\mathcal{C}_G=\bigcup\{\mathcal{C}_k : k=0,1,\ldots\}$ and for $U,V\in\mathcal{C}_G$ put $U<_G V$ iff $U,V\in\mathcal{C}_0$ and $U<V$ or iff $U,V\in(\mathcal{C}_G\setminus\mathcal{C}_0)$ and $U<V$ and if $V\in\mathcal{C}_n$ then $U\in\mathcal{C}_{n+1}$ and $\operatorname{int}(V\setminus U)\neq\emptyset$.
The family $\mathcal{C}_G$ is a pseudobase of G and (a)-(c) hold for $\mathcal{C}_G$ and $<_G$. □

Lemma 4. Let X be a pseudobasically compact space and let $\mathcal{C}$ be a pseudobase of X for which (a)-(c) hold. Then there exists a pseudobase $\mathcal{P}\subseteq\mathcal{C}$ such that $|\mathcal{P}|=\pi w(X)$ and such that (a)-(c) hold for $\mathcal{P}$.

Proof. Observe first, that $\pi w(X)>\omega$. Suppose that $|\mathcal{C}|>\pi w(X)$ and let $\mathcal{B}$ be such a pseudobase of X that $|\mathcal{B}|=\pi w(X)$. For each $B\in\mathcal{B}$ choose, whenever it is possible, $U_B,V_B\in\mathcal{C}$ such that $U_B<V_B$ and $U_B\subseteq B\subseteq V_B$. The family

$\mathcal{P}_1=\{U\in\mathcal{C}$: for some $B\in\mathcal{B}$ we have $U=U_B$ or $U=V_B\}$

is a pseudobase of X and $|\mathcal{P}_1|=\pi w(X)$.

Suppose that we have constructed $\mathcal{P}_k$ for $k\leq n$. For each $P\in\bigcup\{\mathcal{P}_k : k=1,\ldots,n\}$ and $B\in\mathcal{B}$ choose $U_{P,B}\in\mathcal{C}$ such that $U_{P,B}<P$ and $U_{P,B}\subseteq B$ whenever $P\cap B\neq\emptyset$. Put

$\mathcal{P}_{n+1}=\{U\in\mathcal{C}$: there exist $P\in\bigcup\{\mathcal{P}_k : k=1,\ldots,n\}$ and $B\in\mathcal{B}$ such that $U=U_{P,B}\}$.

The family $\mathcal{P}=\bigcup\{\mathcal{P}_n : n=1,2,\ldots\}$ is a pseudobase we require. □

The following is proved in [2].

Lemma 5. Let X be a pseudobasically compact space and let $\pi w(X)$ be smaller than the first measurable cardinal. Let $\mathcal{F}$ be a point finite cover of X consisting of meager sets. If for each $\mathcal{A}\subseteq\mathcal{F}$ the union $\bigcup\mathcal{A}$ has the Baire property, then no non-meager G_δ set can be covered by less than 2^ω elements of $\mathcal{F}$. □

Theorem 1. Let X be a pseudobasically compact space and let $\pi w(X) \leq 2^{\omega}$. If $\mathcal{F}$ is a point finite family of meager sets covering X, then there exists $\mathcal{A} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{A}$ has not the Baire property. □

The theorem of [1] can be reformulated as follows:

Theorem 2. If X is a pseudobasically compact space and $\pi w(X) \leq 2^{\omega}$, then for each map $f: X \to Y$ having the Baire property, where Y is a space with σ-disjoint base, there exists a meager set $F \subseteq X$ such that $f|X \setminus F$ is continuous. □

Using theorems above one can prove easily the following:

Theorem 3 (A. Loveau and S.G. Simpson [4]). Let X be a metric space and $f: [\omega]^{\omega} \to X$ be such a mapping that the counterimage of any open set is completely Ramsey. Then there exists an infinite subset T of ω such that $f([T]^{\omega})$ is separable. □

Theorem 4 (Prikry and Solovay [5]). If X is a metric space and $f: [0,1] \to X$ is a measurable function, then there exists a subset A of [0,1] such that f(A) is separable and the Lebesgue measure of A is equal to 1. □

For details we refer our paper [2].

Let $K^{+}(X)$ denotes the family of all non-void and compact subsets of X. Let $\mathcal{B}(X)$ denotes the family of all subsets of X having the Baire property. A mapping $F: X \to K^{+}(Y)$ is lower $\mathcal{B}(X)$-measurable iff $\{x \in X : F(x) \cap U \neq \emptyset\} \in \mathcal{B}(X)$ for each open $U \subseteq Y$.

Theorem 5. Let X be a pseudobasically compact space, let $\pi w(X) \leq 2^{\omega}$ and let Y be a metric space. Let $F: X \longrightarrow K^{+}(Y)$ be lower $\mathcal{B}(X)$-measurable. Then there exists a $\mathcal{B}(X)$-measurable function $f: X \longrightarrow Y$ such that $f(x) \in F(x)$. □

The theorem above is proved in [3]. We refer to this paper for a detailed discussion of selectors theorems.

References

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