Luděk Zajíček

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ON METRIC PROJECTIONS AND DISTANCE FUNCTIONS IN BANACH SPACES

L.Zajíček

We shall consider a real Banach space X and a nonempty closed subset $F \subset X$. For $x \in X$ denote by $d_F(x)$ the distance from the point x to the set F. The metric projection $P_F(x)$ on the set F is defined as the (possibly) multivalued operator $P_F(x) = \{ y \in F; \|x-y\| = d_F(x)\}$. The set of all x for which $P_F(x)$ contains at least two points will be denoted by A_F. The function is termed δ - convex if it is the difference of two convex functieons. The hypersurface in X is termed Lipschitz(resp. δ-convex) if it is described by a Lipschitz(resp.Lipschitz δ-convex) function(see[7] or [5] and [6]). The sets A_F was studied e.g. in [2],[4],[3],[5]. If X is a separable strictly convex Banach space then A_F can be covered by countably many Lipschitz hypersurfaces [5]. If X is a separable Hilbert space then there exists [1] a convex f_F (namely $f_F(x) = 1/2 (\|x\|^2 - d_F^2(x))$) such that $P_F(x) \subset \partial f_F(x)$. Using a result on the differentiation of convex functions from[6] we immediately obtain the following

<u>Theorem 1.</u> Let X be a separable Hilbert space. Then A_F can be covered by countably many δ-convex hypersurfaces.

<u>Question 1.</u> Let A be a δ-convex hypersurface in R^n. Does there exist F such that $A \subset A_F$?

Note that it is not difficult to prove that a boundary of a convex body in R^n is a subset of an A_F.

Slightly modifying the Asplunds observation concerning the function f_F we can obtain the following theorem.

<u>Theorem 2.</u> Let X be a Hilbert space or a finite dimensio-

nal Banach space such that $\|x\| \in C^2(X-\{0\})$. Then $d_F(x)$ is a locally δ-convex function in X-F.

This theorem has the following consequences.

Theorem 3. Let X be finite dimensional and $\|x\| \in C^2(X-\{0\})$. Then $d_F(x)$ is twice differentiable a.e. in X-F.

Theorem 4. Let X be finite dimensional and $\|x\| \in C^2(X-\{0\})$. Then A_F can be covered by countably many of δ-convex hypersurfaces.

Question 2. For which X each A_F can be covered by countably many of δ-convex hypersurfaces?

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