Jiří Vilímovský
Uniformly continuous selections

In: Zdeněk Frolík (ed.): Abstracta. 8th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1980. pp. 177--179.

Persistent URL: http://dml.cz/dmlcz/701205

8th WINTER SCHOOL (1980)

UNIFORMLY CONTINUOUS SELECTIONS

J. Vilímovský

We refer to [I] for basic concepts pertaining to uniform spaces. Under a space we always mean a Hausdorff uniform space. If $\mathcal{V}$ is a cover of a set X, we put $St^0\mathcal{V} = \mathcal{V}$ and for positive integers n we put $St^n\mathcal{V} = \{St^n(x,\mathcal{V});\ x \in X\}$, where $y \in St^n(x,\mathcal{V})$ iff there are $V_1,\dots,V_n \in \mathcal{V}$ such that $V_i \cap V_{i+1} \neq \emptyset$, $x \in V_1$, $y \in V_n$.

Suppose X,Y are uniform spaces, $F:X \longrightarrow Y$ is a correspondence (a multivalued mapping with at least one value in each point). We shall call F uniformly lower semicontinuous (ulsc), if for every uniform cover $\mathcal{U}$ of Y we can find a uniform cover $\mathcal{V}$ of X such that for all $n \in \omega$ the cover $St^n\mathcal{V}$ refines $F^{-1}(St^n\mathcal{U})$. Under $F^{-1}(A)$ we understand as usual the set $\{x \in X;\ Fx \cap A \neq \emptyset$.

It can be easily proved that if F admits a uniformly continuous selection, then F is ulsc.

We restrict ourselves to Banach spaces Y. At first we present an easy but useful proposition.

<u>Proposition</u>: Let F be a correspondence from a uniform space X into a Banach space Y such that $\bigcup\{Fx; x \in X\}$ is bounded in Y. Then F is ulsc iff $F^{-1}(\mathcal{U})$ is a uniform cover of X for all uniform covers $\mathcal{U}$ of Y.

The first theorem we want to present here is an analogue of Michael's continuous selection theorem (see [M]). A uniform space X is said to have the property (l_1), if for every uniform cover $\mathcal{U}$ of X there is a family $\{\varphi_a\}_{a \in A}$ of nonnegative uniformly continuous functions such that:

(i) $\sum\{\varphi_a(x);\ a \in A\} = 1$ for all $x \in X$

(ii) The mapping $x \mapsto \{\varphi_a(x)\}_{a \in A}$ is uniformly continuous into $l_1(A)$

(iii) The family $\{coz\, \varphi_a\}_{a \in A}$ refines $\mathcal{U}$.

The examples of spaces having the property (l_1) are for instance topologically fine spaces or spaces having a basis of finite-dimensional covers. On the other hand no infinite-dimensional normed space has the property (l_1) (see [Z]).

Theorem S1: Suppose X is a uniform space with property (l_1), Y is a Banach space, $F:X \longrightarrow Y$ is a ulsc correspondence such that all Fx are convex and $\bigcup\{Fx;\ x \in X\}$ is bounded. Then for every $\varepsilon > 0$ there is a uniformly continuous mapping $\varphi : X \longrightarrow Y$ such that $\mathrm{dist}(\varphi(x), F(x)) < \varepsilon$ for all $x \in X$.

Theorem S1 gives something less than a selection, but it can be shown that even if Fx are closed and all other assumptions of S1 are fulfiled, the selection neednot exist. On the other hand if we restrict ourselves to some special Banach spaces Y, we obtain better selection theorems.

Theorem S2: ([PV]) If F is a closed-convex valued ulsc correspondence from an arbitrary uniform space into the real line, then F admits a uniformly continuous selection.

Theorem S3: Suppose X is a uniform space, Y is a Banach space of type C(K), where K is compact extremally disconnected, $F:X \longrightarrow Y$ is a ulsc correspondence such that all Fx are closed balls (with possibly infinite radii). Then F admits a uniformly continuous selection.

Now we mention several applications of selection theorems. Having a uniformly continuous mapping f on a subspace A of X ranging in a Banach space Y, to find a uniformly continuous extension of f over X means nothing more than to find a uniformly continuous selection from the correspondence

$$F(x) = \begin{cases} \{f(x)\} & \text{if } x \in A \\ Y & \text{if } x \notin A \end{cases}$$

Therefore we obtain immediately from S1-S3 the following extension theorems:

Theorem E1: Suppose X has the property (l_1), f a uniformly continuous bounded mapping from a subspace of X into a Banach space. Then f can be uniformly extended over X.

This theorem was proved firstly by P.Pták using standart partitions of unity technique, the first example of a Banach space such that not every bounded uniformly continuous mapping ranging in it can be extended was constructed by J Lindenstrauss.

Theorem E2: Let A be a subspace of a uniform space X and let

$h: A \longrightarrow Y$ be a uniformly continuous mapping into a Banach space of type C(K) with K compact extremally disconnected. Then h can be uniformly extended to X iff for every uniform cover $\mathcal{U}$ of Y there is a uniform cover $\mathcal{V}$ of X such that $h^{-1}(St^n \mathcal{U})$ is refined by $St^n \mathcal{V} \upharpoonright A$ for all $n \in \omega$.

Another direct application is the following approximation problem: If f is a Banach-valued mapping on a uniform space, find the best approximation of f by a uniformly continuous one. For example Hahn [H] proved that if f is a real-valued function on a uniform space X, $\varepsilon > 0$ and $f^{-1}(\mathcal{S}(\varepsilon))$ is a uniform cover of X ($\mathcal{S}(\varepsilon)$ is the cover consisting of all balls with radii ε), then there is a uniformly continuous function φ such that $|fx - \varphi x| \leq \varepsilon$ for all $x \in X$. We are able to prove directly from S2 even much more

<u>Theorem</u>: Let f be a real valued function on a uniform space X. Put $\varepsilon_0 = \inf \{ \varepsilon > 0 \mid f^{-1}(\mathcal{S}(\varepsilon))$ is uniform$\}$. Then there is a uniformly continuous function φ such that $|f(x) - \varphi(x)| \leq \varepsilon_0$ for all $x \in X$.

Similar results we may obtain also from S1,S3, the only thing which is to prove in all the cases is to prove that the correspondence assigning to every point the ball with center f(x) and radius ε_0 is ulsc.

<u>References:</u>

[H] Hahn H.: Über halbstetige und unstetige Funktionen, Sitzungsb. Akad.Wien 126(1917), 91-110

[I] Isbell J.R.: Uniform spaces, Amer.Math.Soc.,Providence 1964

[M] Michael E.: Continuous selections I, Ann.of Math. 63(1956), 361-382

[PV] Preiss D., Vilímovský J.: In-between theorems in uniform spaces, Trans.Amer.Math.Soc., to appear

[Z] Zahradník M.: l_1-partitions of unity on normed spaces, Seminar Uniform Spaces 1973-74, MÚ ČSAV Prague 1975, 201-216