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Let $\mathcal{V}$ be a non-void class of algebras of a joint type $\mathcal{T}$. The famous Birkhoff variety theorem [1] states: $\mathcal{V}$ is a variety iff it is definable by equations for polynomial symbols. The former means that $\mathcal{V}$ is closed under formation of products, subalgebras and homomorphic images. The latter means that there is a family E of equations $p=q$, where p, q are polynomial symbols, such that an algebra A of the type $\mathcal{T}$ belongs to $\mathcal{V}$ iff for each equation $p=q$ in E the induced operations p_A, q_A coincide.

Our aim is to formulate Birkhoff theorem for finite algebras. Recall that, e.g., finite groups, viewed as algebras with one binary operation, cannot be defined by equations in polynomial symbols in spite of being closed under formation of finite products, subalgebras and homomorphic images. The classes of finite algebras with the latter property will be called varieties of finite algebras. We shall characterize them by equations for implicit operations.

An n-ary implicit operation in a class $\mathcal{A}$ of algebras of a type $\mathcal{T}$ is a rule π assigning to each algebra $A \in \mathcal{A}$ an n-ary operation $\pi_A : A^n \longrightarrow A$ in a way, compatible with each homomorphism $h : A \longrightarrow B$ in the natural sense: $\pi_B(ha_1,\dots,ha_n) = h\,\pi_A(a_1,\dots,a_n)$ for all $a_1,\dots,a_n \in A$. Clearly, each polynomial symbol induces an implicit operation; implicit operations arising in this way will be called explicit. Thus, in the Birkhoff theorem above, polynomial symbols can be replaced by explicit operations. Moreover, if $\mathcal{a}$ is the class of all (both finite and infinite) algebras of a given type then, by Lawvere [3] , implicit and explicit operations coincide. Hence our main result is a direct finite analogue of the Birkhoff theorem:

Theorem. Let $\mathcal{a}$ be a variety of finite algebras of a finite (finitary) type. A non-void class $\mathcal{V} \subset \mathcal{a}$ is a variety of finite algebras iff it is definable by equations for implicit operations in $\mathcal{a}$.

Example 1. Groups. In the variety $\mathcal{a}$ of finite monoids define a unary implicit operation π as follows. Given a finite monoid $A = (X,\cdot,1)$ and an $x \in X$, let $\pi_A(x)$ be the only idempotent contained in the subsemigroup of A generated by x . Then the equation $\pi(x) = 1$ defines the variety of finite groups.

Example 2. Permutations. Let $\mathcal{a}$ be the class of all finite algebras with one unary operation f . For every algebra $A = (X,f_A)$ let $\pi_A = f_A^n$ where n is the least non-negative integer such that f_A^n is idempotent. Then π is an implicit operation [4] . The equation $\pi(x) = x$ defines the variety of permutations, i.e. finite algebras A with f_A bijective.

Remark. Another characterization of varieties of finite algebras was given by Eilenberg and Schützenberger [2] : Every variety $\mathcal{V}$ of finite algebras of a finite type is defined ultimately by a sequence $p_n = q_n$ of equations for polynomial symbols in the sense that an algebra belongs to $\mathcal{V}$ iff it satisfies the equation $p_n = q_n$ for all sufficiently large n . Our approach was inspired by the work of Rosický and Polák [4] who showed that finite algebras admit many implicit operations which are not explicit.

The proof of our theorem utilizes topological methods which we are going to sketch here. Let $\bar{\Omega}_n$ denote the set of all n-ary implicit operations in $\mathcal{A}$. With each algebra $A \in \mathcal{A}$, let us associate a pseudometric d_A on $\bar{\Omega}_n$ where

(1) $d_A(\pi, \varrho) = 0$ if $\pi_A = \varrho_A$, $d_A(\pi, \varrho) = 1$ otherwise.

Since isomorphic algebras induce the same pseudometrics, the family $\{d_A;\ A \in \mathcal{A}\}$ is countable and hence we can make it into a sequence $\{d_k\}$ which induces a single pseudometric, in fact, a metric

$$\sum_k 2^{-k} d_k \ . \tag{2}$$

This makes $\bar{\Omega}_n$ a metric space.

Lemma. $\bar{\Omega}_n$ <u>is compact.</u>

<u>Remark.</u> The set Ω_n of explicit operations is dense in $\bar{\Omega}_n$. Hence implicit operations can be topologically reconstructed from the explicit ones: $\bar{\Omega}_n$ is a completion of Ω_n.

Now if a class of finite algebras is defined by equations for implicit operations then it is obviously a variety of finite algebras. Conversely, let $\mathcal{V}$ be a variety of finite algebras. Define a pseudometric $d_{\mathcal{V}}$ on $\bar{\Omega}_n$ by the sum (2) restricted to summands corresponding to algebras in $\mathcal{V}$.

Lemma. <u>An algebra A belongs to $\mathcal{V}$ iff the map id : $(\bar{\Omega}_n, d_{\mathcal{V}}) \longrightarrow (\bar{\Omega}_n, d_A)$ is continuous for every</u> n.

Let E be the family of all equations for implicit operations which are satisfied by all algebras in $\mathcal{V}$. If an algebra $A \in \mathcal{A}$ satisfies all equations in E then $A \in \mathcal{V}$ by the following lemma applied to $\delta = d_{\mathcal{V}}$, $\sigma = d_A$.

Lemma. Let M be a compact metric space and let δ, σ be continuous pseudometrics on M. Then the map id : $(M,\delta) \longrightarrow (M,\sigma)$ is continuous iff $\delta(x,y) = 0$ implies $\sigma(x,y) = 0$ for all $x,y \in M$.

References

[1] G. Birkhoff: On the structure of abstract algebras. Proc. Cambridge Phil.Soc. 31(1935), 433-454

[2] S. Eilenberg and M.P. Schützenberger: On pseudovarieties, Advances in Math. 19(1976), 413-418

[3] F.W. Lawvere: Functorial semantics of algebraic theories. Proc.Nat.Acad.Sci., 50(1963), 869-872

[4] J. Rosický and J. Polák: Definability on finite algebras; manuscript