Svatopluk Poljak; Daniel Turzík

Weak-join of matriods

In: Zdeněk Frolík (ed.): Abstracta. 8th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1980. pp. 130--133.

Persistent URL: http://dml.cz/dmlcz/701193

EIGHTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1980)

Weak-join of matroids

S. Poljak, D. Turzík

This note deals with glueing of matroids. Some constructions in the matroid theory can be considered as glueing, e.g. the sum of matroids, the simultaneous extension [2], the Dilworth truncation [3]. The presented approach aims to applications in Ramsey theory, for particular results see [4]. We introduce the notion of the weak-join which, if it exists, is the free-est amalgamation. We give a sufficient condition for the existence of a weak-join.

A matroid $M(X)$ is a set X with a rank function $r_M: \mathcal{P}(X) \longrightarrow N$ satisfying:

1/ $r(\emptyset) = 0$

2/ $r(\{x\}) \leq 1$ for $x \in X$

3/ $r(A) \leq r(B)$ for $A \subseteq B \subseteq X$

4/ $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ for $A, B \subseteq X$

A matroid M is called modular if

$r(F) + r(G) = r(F \cup G) + r(F \cap G)$ for every pair F, G of flats.

For $A \subseteq X$ the restriction of M to the subset A is denoted by $M|A$.

Definition 1: Let $\mathcal{H} = (V, (E_i \mid i \in I))$ be a hypergraph and $\mathcal{M} = (M_i(E_i) \mid i \in I)$ be a system of matroids. A matroid $M(V)$ is called an amalgamation of $\mathcal{M}$ with respect to $\mathcal{H}$ if $M \mid E_i = M_i(E_i)$.

A. Weak-join of two matroids

If $M(X_1 \cup X_2)$ is an amalgamation of matroids $M_1(X_1)$, $M_2(X_2)$ then

$$r_M(A) \leq r_1(B \cap X_1) + r_2(B \cap X_2) - r_1(B \cap X_1 \cap X_2)$$

for every $A \subseteq B \subseteq X_1 \cup X_2$.

<u>Definition 2</u>: Let $M_1(X_1)$, $M_2(X_2)$ be matroids with $M_1|X_1 \cap X_2 = M_2|X_1 \cap X_2$. Put

$$\rho(A) = r(A \cap X_1) + r(A \cap X_2) - r(A \cap X_1 \cap X_2) \text{ and}$$

$$R(A) = \min\{\rho(B) \mid A \subseteq B\} \quad \text{for } A \subseteq X_1 \cup X_2.$$

If R is a rank function then the matroid $M(X_1 \cup X_2)$ defined by R is called a <u>weak-join</u> of M_1 and M_2.

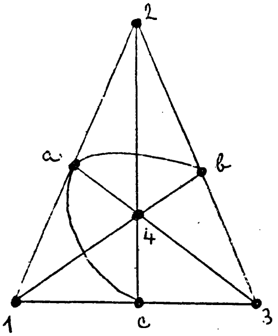
<u>Theorem 1:</u> Let $M_1(X_1)$, $M_2(X_2)$ be matroids with $M_1|X_1 \cap X_2 = M_2|X_1 \cap X_2$. Then

(i) If a weak-join exists then it is an amalgamation of M_1 and M_2.

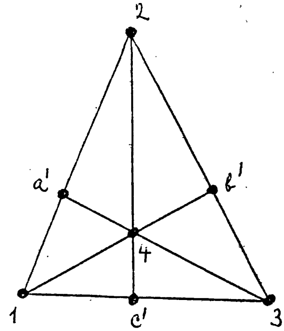
(ii) If a weak-join $M(X_1 \cup X_2)$ exists and $N(X_1 \cup X_2)$ is another amalgamation then the identity mapping $i: M \longrightarrow N$ is a weak map, i.e. weak-join is the free-est amalgamation (with respect to weak maps).

(iii) If the matroid $M_1|X_1 \cap X_2$ is modular then the weak-join of M_1 and M_2 exists.

<u>Example:</u> The smallest non-modular matroid C_4 is that formed by four points in general position in the plane. The following picture gives an example of two matroids which intersect in C_4 and which have no amalgamation.



Fano



Fano^-

In [1] Brylawski introduced a notion of a strong-join, which is, if it exists, the free-est amalgamation with respect to strong maps. Let us remark that any strong-join is the weak-join.

B. Weak-join with respect to tree

Definition 3: A hypergraph $(V,(E_i \mid i \in I))$ is called a *tree-hypergraph* if there exists a graph $\mathcal{T} = (I,T)$ which is a tree such that: if the vertex i lies on the path between vertices j and k in $\mathcal{T}$ then $E_j \cap E_k \subseteq E_i$.

The following definition is a generalization of the weak-join to the tree-hypergraph.

Definition 4: Let $\mathcal{H} = (V,(E_i \mid i \in I))$ be a tree-hypergraph and $\mathcal{M} = (M_i(E_i) \mid i \in I)$ be a system of matroids satisfying $M_i|E_i \cap E_j = M_j|E_i \cap E_j$ for every $i,j \in I$. Put

$$\varrho(A) = \sum_{i\in I} r(A \cap E_i) - \sum_{i \neq j \in I} r(A \cap E_i \cap E_j) + \ldots + (-1)^{|I|+1} r\Big(A \cap \bigcap_{i\in I} E_i\Big)$$

$$R(A) = \min\{\varrho(B) \mid A \subseteq B\} \quad \text{for} \quad A \subseteq V.$$

If R is a rank function then the matroid $M(V)$ defined by R is called a weak-join of $\mathcal{M}$ with respect to $\mathcal{H}$.

<u>Theorem 2</u>: Let $\mathcal{H}$ and $\mathcal{M}$ be as in the above definition. Then

(i) If the weak-join of $\mathcal{M}$ with respect to $\mathcal{H}$ exists then it is the free-est amalgamation.

(ii) If $M_i | E_i \cap E_j$ is modular for every $i, j \in I$, $i \neq j$ then the weak-join of $\mathcal{M}$ with respect to $\mathcal{H}$ exists.

<u>References:</u>

[1] T.Brylawski: Modular construction for combinatorial geometries, Trans.Amer.Math.Soc. 203(1975),1-44

[2] H.Crapo, G.C.Rota: On the foundations of combinatorial theory: Combinatorial Geometries, M.I.T. Press, Camb. Mass. (1970)

[3] J.H.Mason: Matroids as the study of geometrical configurations, In: Higher combinatorics (ed. M.Aigner), D.Reidel Publ.Co., Dordrecht, Boston(1977), 133-178

[4] J.Nešetřil, S.Poljak, D.Turzík: Amalgamation of matroids and its applications, to appear in JCT(B)