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The Use of Mixed Norms: Two Examples

by

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In a paper on "the Feynman integral" G.W. Johnson and D.L. Skong [JS1] found a certain "mixed norm" useful. Once oriented in this direction they have found mixed norms useful in several places in their work on the Feynman integral and related problems [JS2-JS6]. In a paper on thin sets in harmonic analysis, specifically, on "p-Sidon sets", G.W. Johnson and G.S. Woodward [JW] proved a certain mixed-norm inequality which was crucial for the results of that paper. In the two examples which will be discussed, mixed norms were not involved in the statement of the problem but they were involved in the solution.

My experience with the use of mixed norms has led me to the strong conviction that the mixed-norm point of view could be used to advantage in many more situations if more mathematicians thought in these terms. In some cases it seems to enable one to solve problems that could not otherwise be solved.

G.W. Johnson and L.V. Petersen [P and JP] have attempted to give the beginnings of a general framework for considering mixed norm results. W.A.J. Luxemburg [L] has independently developed much the same theory; indeed a more satisfactory form of the key theorem will appear in [L].

<u>1st Example:</u> Let $C_o[0,t]$ denote the real-valued continuous functions X on $[0,t]$ such that $X(0) = 0$. Given a function $F : C_o[0,t] \to \mathbb{C}$ (the complex numbers) one can define $J(F)$, the analytic operator-valued Feynman integral of F [CS].

If it exists, $J(F)$ is a bounded linear operator on $L_2(\mathbb{R})$.

Let $0 < t_1 < \dots < t_n \leq t$ $(n \geq 3)$ and let $f : \mathbb{R}^n \to \mathbb{C}$. Let $F(X) = f(X(t_1), \dots, X(t_n))$ for X in $C_0[0,t]$. Johnson and Skoug sought some L_p condition on f insuring the existence of $J(F)$ but instead found

Theorem: [JS1] There exists f in $\bigcap_{1 \leq p \leq \infty} L_p(\mathbb{R}^n)$ such that $J(F)$ fails to exist.

On the other hand, they proved.

Theorem: [JS1] If $\| f \|_2 \| \dots \|_2 < \infty$, then $J(F)$ exists where

$$\| f \|_2 \| \dots \|_2 = \left\{ \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \overset{(n-2)}{\cdots\cdots} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} | f(u_1, \dots, u_n) |^2 du_1 \right)^{1/2} \right. \right.$$
$$\left. \left. . du_2 \dots du_{n-1} \right]^2 du_n \right\}^{1/2} .$$

2nd Example: Let $1 \leq p < 2$. A subset E of the integers is said to be p-Sidon $(E \in S_p)$ if there exists a constant C such that for every finite subset F of E and any subset $\{c_n : n \in F\}$ of complex numbers indexed by F we have

$$\left(\sum_{n \in F} |c_n|^p \right)^{1/p} \leq C \left\| \sum_{n \in F} c_n e^{-int} \right\|_\infty .$$

(See [ER, LR, JW].) If $1 \leq p_1 < p_2 < 2$, clearly $S_{p_1} \subset S_{p_2}$.

In the deepest theorem in the paper [ER] it is shown that

Theorem: [ER] If $1 \leq p_1 < 4/3 \leq p_2 < 2$, then $S_{p_1} \subsetneq S_{p_2}$.

A key element in the argument was a mixed norm inequality of Littlewood [Li]:

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^2 \right)^{1/2} + \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}|^2 \right)^{1/2} \geq \left(\sum_{i,j} |a_{ij}|^{4/3} \right)^{3/4} .$$

[JW] extended the result of [ER]. One of the two substantial difficulties was finding and proving the following extension of Littlewood's inequality.

$$\sum_{k=1}^{n} \sum_{i_k} \Big(\sum_{(i_k)} |a_{i_1,\dots,i_n}|^2 \Big)^{1/2} \geq \Big(\sum_{i_1,\dots,i_n} |a_{i_1,\dots,i_n}|^{\frac{2n}{n+1}} \Big)^{\frac{n+1}{2n}} .$$

[(i_k) indicates summation over all the indices except the k^{th}.]

Theorem: [JW] If $1 \leq p_1 < 4/3 \leq p_2 < 6/4 \leq \dots \leq p_{n-1} < \frac{2n}{n+1} \leq p_n < \dots$ then $S_{p_1} \subsetneq S_{p_2} \subsetneq \dots \subsetneq S_{p_{n-1}} \subsetneq S_{p_n} \subsetneq \dots$.

Recently R. Blei [B] has shown that $S_{p_1} \subsetneq S_{p_2}$ whenever $1 \leq p_1 < p_2 < 2$. His argument involves a further extension and refinement of the old mixed norm inequality of Littlewood.

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