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Realizing Homomorphisms of Category Algebras

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In a series of papers D. Maharam and A.H. Stone investigated the problem of realizing isomorphisms and automorphisms of category algebras by certain point-mappings. It is the purpose of this talk to present a similar result for homomorphisms of category algebras.

Definitions:

For a topological space Z let $\mathfrak{k}_1(Z)$ denote the σ-ideal of all sets of first category in Z, $\mathfrak{B}_p(Z)$ the σ-field $\{B \subset Z \mid \exists\, U \text{ open: } B \triangle U \in \mathfrak{k}_1(Z)\}$ of sets with the Baire property, and $\mathcal{L}(Z) = \mathfrak{B}_p(Z)/\mathfrak{k}_1(Z)$ the category algebra of Z. The symbol $\mathfrak{B}(Z)$ always stands for the Borel field of Z.

If Z is a Baire topological space (i.e. if an open subset U of Z belongs to $\mathfrak{k}_1(Z)$ if and only if $U = \emptyset$) then, for every $C \in \mathcal{L}(Z)$, there exists exactly one regular open set $\theta(C)$ in Z (i.e. $\theta(C) = \overset{\circ}{\overline{\theta(C)}}$) which is contained in the equivalence class C (i.e. $[\theta(C)] = C$).
The map $\theta: \mathcal{L}(Z) \to \mathfrak{B}(Z)$ has the following properties:

(i) $\theta([\emptyset]) = \emptyset$, $\theta([Z]) = Z$

(ii) $\theta(C \cap D) = \theta(C) \cap \theta(D) \qquad \forall\, C,D \in \mathcal{L}(Z)$

(iii) $\theta(\bigvee\{C_i \mid i \in I\}) = \overset{\circ}{\overline{\bigcup\{\theta(C_i) \mid i \in I\}}}$ for every family $(C_i)_{i \in I}$ in $\mathcal{L}(Z)$.

Theorem:

Let X be a complete metric space, Y a Baire topological space, and $\Phi: \mathfrak{B}(X) \to \mathcal{L}(Y)$ a σ-homomorphism. Then the following properties of Φ are equivalent:

(i) For every family $(U_i)_{i \in I}$ of open subsets of X the identity

$$\Phi(\bigcup\{U_i \mid i \in I\}) = \bigvee\{\Phi(U_i) \mid i \in I\}$$

holds.

(ii) For every open covering $(U_i)_{i\in I}$ of X (with card $I \leq$ wt X)

$$\bigvee \{\Phi(U_i) \mid i\in I\} = [Y].$$

(iii) There exists a dense G_δ-set $D \subset Y$ and a continuous map $f\colon D \to X$ with $[f^{-1}(B)] = \Phi(B)$ for all $B \in \mathcal{B}(X)$.

<u>Proof:</u> (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (i) are easy to check.
(ii) $\Longrightarrow$ (iii): For $y \in Y$ let $\mathcal{F}_y = \{B \in \mathcal{B}(X)\colon y \in \theta(\Phi(B))\}$. Let D be the set $\{y \in Y \mid \mathcal{F}_y$ converges$\}$ and define $f\colon D \to X$ by $f(y) = \lim \mathcal{F}_y$. Then (D,f) has the required properties.

<u>Corollary 1:</u>

Let X be a complete metric space, $m =$ wt X, Y a Baire topological space, and $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ a σ-homomorphism. Then the following conditions are equivalent:

(i) Φ is an m-homomorphism.

(ii) Φ is a complete homomorphism.

(iii) There exists a dense G_δ-set $D \subset Y$ and a continuous map $f\colon D \to X$ with $[f^{-1}(B)] = \Phi([B])$ for all $B \in \mathcal{B}_p(X)$.

<u>Proof:</u> The corollary is obtained from the theorem by considering the σ-homomorphism $\mathcal{B}(X) \to \mathcal{L}(Y)$, $B \mapsto \Phi([B])$.

<u>Corollary 2:</u> (Maharam - Stone [3])

Let X and Y be complete metric spaces and $\Phi\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ an isomorphism onto. Then there exist dense G_δ-sets $D \subset Y$ and $E \subset X$ and a homeomorphism f from D onto E such that $[f^{-1}(A)] = \Phi([A])$ and $[f(B)] = \Phi^{-1}([B])$ for all $A \in \mathcal{B}_p(X)$, $B \in \mathcal{B}_p(Y)$.

<u>Corollary 3:</u>

Let X be a separable metric space, Y a Baire topological space, and $f\colon Y \to X$ an arbitrary mapping. Then f is $\mathcal{B}_p(Y)$-$\mathcal{B}(X)$-measurable if and only if there exists a dense G_δ-set $D \subset Y$ such that $f_{|D}$ is continuous.

Corollary 4: (Fort [1]) (cf. Namioka [6], Theorem 1.2)

Let Y be a Baire topological space, Z a locally compact separable metric space, and R a separable metric space. If $f: Y \times Z \to R$ is continuous in each variable separately then there exists a dense G_δ-subset D of Y such that f is continuous at each point of $D \times Z$.

Let us recall that a finite measure ν on the Borel field $\mathfrak{B}(X)$ of a topological space X is called τ-continuous if, for every filtering decreasing family $(A_i)_{i \in I}$ of closed sets in X with $\bigcap_{i \in I} A_i = \emptyset$, the equality $\inf\{\nu(A_i) \mid i \in I\} = 0$ is satisfied. For a measure space $(Y, \mathfrak{A}, \mu)$ let $\mathfrak{A}/\mu$ be the quotient of the σ-field $\mathfrak{A}$ w.r.t. the σ-ideal of μ-nullsets.

Corollary 5:

Let X be a complete metric space, $(Y, \mathfrak{A}, \mu)$ a complete finite measure space, and $\Phi: \mathfrak{B}(X) \to \mathfrak{A}/\mu$ a σ-homomorphism such that $\mu \circ \Phi$ is a τ-continuous measure on $\mathfrak{B}(X)$. Then there exists an $\mathfrak{A}$-$\mathfrak{B}(X)$-measurable map such that $[f^{-1}(B)] = \Phi(B)$ for all $B \in \mathfrak{B}(X)$.

Proof: It follows from Ionescu Tulcea [2], p. 54, Proposition 1 that there exists a topology $\mathfrak{T}$ on Y such that $(Y, \mathfrak{T})$ is a Baire topological space, $\mathfrak{A}$ equals the σ-field of sets with the Baire property w.r.t. $\mathfrak{T}$, and the μ-nullsets are just the sets of first category in $(Y, \mathfrak{T})$. Thus $\mathfrak{A}/\mu$ is the category algebra of $(Y, \mathfrak{T})$. Using the τ-continuity of $\mu \circ \Phi$ we obtain immediately that Φ satisfies condition (ii) in the theorem. Hence the theorem implies the existence of an $\mathfrak{A}$-$\mathfrak{B}(X)$-measurable map with the desired properties.

References

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