Martin Gavalec; Peter Vojtáš
Ramsey-type theorems

In: Zdeněk Frolík (ed.): Abstracta. 8th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1980. pp. 79--81.

Persistent URL: http://dml.cz/dmlcz/701181

EIGHTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1980)

RAMSEY-TYPE THEOREMS

Martin Gavalec, Peter Vojtáš

I. Rival and B. Sands proved in [1] the following two modifications of the well-known Ramsey theorem.

RS 1. Every infinite graph G contains an infinite subgraph H such that every vertex of G is adjacent to precisely, none, one, or infinitely many, of the vertices of H .

RS 2. Every infinite partially ordered set P of finite width contains an infinite chain C such that every element of P is comparable with none, or infinitely many, of the elements of C .

A natural question is, whether these theorems can be generalized to higher cardinalities. It is known that the generalization of Ramsey theorem is valid only for weakly compact cardinals. In contrary, the generalization of theorem RS 1 holds for all regular cardinals. For singular cardinals we have a weaker version.

Theorem 1. If $\varkappa$ is an infinite regular cardinal, then every graph G of cardinality $\geq \varkappa$ contains a subgraph H of cardinality $\geq \varkappa$ such that every vertex of G is adjacent to precisely, none, one, or $\geq \varkappa$ many, of the vertices of H .

A subgraph H with the property described in theorem 1 will be called a $(0, 1, \varkappa)$-subgraph of G .

<u>Theorem 2</u>. If $\varkappa$ is an infinite singular cardinal, then every graph G of cardinality $\varkappa$ contains, for any $\alpha < \varkappa$, a $(0, 1, \alpha)$-subgraph H of cardinality $\varkappa$.

Remark. An easy construction gives an example showing that theorem 2 is the best result for $\varkappa$ singular.

Theorem RS 2 can be generalized for regular cardinals, but not for singular ones.

A chain C in a partially ordered set P will be called a <u>$(0,\varkappa)$-chain</u> in P if C is non-empty and if every element of P is comparable with precisely, none or $\geq \varkappa$ many, of the elements of C. Clearly, a $(0,\varkappa)$-chain is of cardinality $\geq \varkappa$.

<u>Theorem 3</u>. If $\varkappa$ is an infinite regular cardinal, then every partially ordered set P of cardinality $\varkappa$ and of finite width n contains a $(0,\varkappa)$-chain.

<u>Theorem 4</u>. If $\varkappa$ is an infinite singular cardinal, then there is a partially ordered set P of cardinality $\varkappa$ and of width 3 such that there are no $(0,\varkappa)$-chains in P.

Remark. The assumption of finite width n in theorem 3 cannot be weakened, not even to width ω. For any infinite regular cardinal $\varkappa$ a partially ordered set P of cardinality $\varkappa$ can be constructed such that any antichain in P is finite (i.e. the width of P is ω) and there are no $(0,\varkappa)$-chains in P.

If we restrict ourselves to trees, instead of arbitrary partially ordered sets, then we get stronger results.

<u>Theorem 5</u>. If $\varkappa$ is an infinite regular cardinal, then every tree T of cardinality $\varkappa$ and of width $<\varkappa$ contains a $(0,\varkappa)$-chain.

Theorem 6. If $\varkappa$ is an infinite regular cardinal, then the assertion that "every tree T of cardinality $\varkappa$ and of width $\varkappa$ contains a $(0, \varkappa)$-chain" is equivalent with Suslin's hypothesis on $\varkappa$.

Reference

[1] I.Rival, B.Sands: On the adjacency of vertices to the vertices of an infinite subgraph, Univ. of Calgary, Canada, Research paper No. 424, April 1979.